The Frame–Budget Approach (FBA)

How Time, Dynamics, and Geometry Emerge from Budget Flows

An Operational Bridge between Quantum Mechanics and General Relativity

Part VIII: Classical Limit, Thermodynamics & Aging

Dipl. Wirt.-Inf. Jens Tetzner

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Part VIII
Classical Limit, Thermodynamics & Aging

VIII.1 Introduction & Target Picture

VIII.1.1 Motivation

The Frame–Budget Approach (FBA)[1] aims in this part at a *consistent* derivation of two transitions that are often treated separately in standard accounts: (i) the *classical limit* for (typically) open quantum systems and (ii) the *thermodynamic arrow* in the unselected picture. The common core is not an additional dynamical assumption, but the combination of admissible microdynamics (CPTP/GKLS), well-defined coarse-graining, and the resulting information monotonicities (DPI/Spohn). This makes precise *in which regimes* an effective classical description is admissible, and why irreversibility in the effective picture appears as a consequence of admissible processing (rather than as a heuristic extra postulate).[2] [3] [4]

We deliberately distinguish (a) a *mathematical limit* (e.g. continuum/large-N/scale limits) from (b) an *operational limit* (measurement resolution, protocol classes, admissible coarse-grainings): only the operational layer makes statements about "classically observable" robust in finite practice.

Working notion: "classical" in the FBA

In Part VIII, "classical" does not mean "$\hbar \to 0$ by postulate", but: there exists an admissible coarse-graining channel $\mathcal{R}$ (coarse-graining) such that $p = \mathcal{R}(\rho)$ provides an effective (pointer-/protocol-stable) state description whose predictions are robust under further admissible processing (DPI/Spohn). The mathematical limit (continuum/large-N/scales) serves only to derive *closed* classical equation forms (e.g. Fokker–Planck/Langevin) as a limiting description; which quantities are "classically observable" is decided by the operational layer.

Concretely, we show: From CPTP/GKLS dynamics *together with* admissible coarse-graining (decoherence, pointer selection, Markov closure), *in suitable limiting and closure regimes* one obtains effective classical master, Fokker–Planck, and Langevin equations. Entropy production and second-law statements are identified as consequences of the data-processing arrow (Spohn/DPI) in the *unselected* picture (under clearly stated reference/Markov assumptions, respectively). Fluctuation theorems (IFT/Crooks/Jarzynski) are then formulated as *pathwise versions*, but only under the explicitly stated additional assumptions (microreversibility / local detailed balance and suitable boundary conditions). Building on this, we define *aging* as the calibrated accumulation of *irreversible internal* budget usage along a worldline and distinguish it from mere proper-time accumulation, because only then does one obtain an empirically sharp measure of irreversible "system history" that cannot be confused with

[1] An overview of all parts of the FBA treatise, including download links, can be found in Section VIII.12 of this document.

[2] See FBA Part I: FBA – Foundations, Secs. I.1–I.6.

[3] See FBA Part IV: Dynamics & Measurement (GKLS), Secs. IV.2–IV.7.

[4] See FBA Part II: Time, Proper Time & Minkowski Geometry, in particular Sec. II.4 as well as Secs. II.6–II.7.

kinematics (time dilation). (Entropies S are treated as information-theoretic, *dimensionless* quantities; physical entropies carry the factor k_B where appropriate, and $\beta = (k_B T)^{-1}$ is used only when T has been introduced operationally.)

VIII.1.2 Logical path of the treatise

The structure is chosen so that the chain from microstructure to macroscopic description contains no hidden jumps: first the admissible operations and their monotonicities are fixed; only then are classical equations and thermodynamic statements extracted as consequences of these fixations (plus clearly marked closure/calibration assumptions). Otherwise, classical dynamics and the second law would look like additional postulates rather than consequences of admissible processing.

1. *Sequence & budget:* The ordered sequence of global states with minimal events carries a budget calculus (internal/external/irreversible). This is the starting point, because without balance and decomposition structure neither "costs" nor "irreversibility" can be defined operationally.

2. *Admissible dynamics:* On the micro level, processes are CPTP; in the time-continuum we use (time-local) GKLS descriptions in the unselected picture (more precisely: such effective generators only in regimes in which a CP-divisible, time-local description is operationally justified) and we work with thermally motivated references (stationary or instantaneous, depending on protocol class and the existence of a suitable reference). This layer is necessary because we want to derive the classical limit from allowed microdynamics, not from classical assumptions.

3. *Coarse-graining:* An admissible coarse-graining channel $\mathcal{R}$ (operationally pointer-stable in the respective protocol) produces effective classical states $p = \mathcal{R}(\rho)$; DPI/Spohn yields contractions/monotonicities for admissible processing. This step is the mechanism that explains "effective classicality": classical states are the stable carriers of the remaining information, robust under admissible coarse-graining.

4. *Classical equations:* From the master equation one obtains, in a scale limit, Kramers–Moyal / Fokker–Planck and equivalent Langevin forms; the reversible part yields, in a suitable limiting regime, Liouville/Hamilton dynamics (Ehrenfest limit). This makes visible why different classical equation forms do not compete, but are different projections of the same coarse-graining picture.

5. *Thermodynamic arrow:* Entropy production Σ is nonnegative in the unselected GKLS picture, *provided* the respective reference/stationarity structure (Spohn setting) is satisfied; Clausius and Landauer inequalities follow as budget-calibrated consequences. Fluctuation theorems (IFT/Crooks/Jarzynski) connect to this, but only under explicit path/balance assumptions. This step is the actual "arrow" statement: irreversibility appears as a monotonicity under admissible processing, not as an extra postulate.

6. *Aging:* A is defined as a calibrated line integral of the *irreversible internal* budget rate along a worldline and is thus conceptually and empirically separable from geometric proper time τ_{geo}. This separation is crucial because only A captures the irreversible component, while τ_{geo} also accumulates in reversible regimes.

VIII.1.3 Scope and delimitation

We work *flat* (no curvature term) and *kinematically* (no backreaction). Dynamics is GKLS-conform, or time-locally CP-divisible; non-Markovian extensions serve only for comparison and are not used as a source of monotonicity. Locality/no-signalling are implemented as admissibility conditions via the composition structure introduced in the FBA – Foundations and the bounds fixed there "no budget inflation by rewiring" (no additional dynamical assumption in this part, but part of the admissibility framework).[5] Questions of gravitation from budget flows, scales/renormalization, and cosmic dynamics are developed in Parts VI, VII, and IX.[6] [7] [8]

VIII.1.4 Contribution relative to standard approaches

Instead of (i) introducing the classical limit via ad-hoc "classical" postulates or (ii) justifying the second law heuristically, we set up the derivation such that the essential statements follow *from* CPTP/GKLS, DPI/Spohn, and the budget calculus, while the required additional assumptions (pointer stability, Markov closure, detailed balance or microreversibility, boundary conditions) are stated explicitly. Pointer structures and Fokker–Planck coefficients appear as the operationally most stable modes of admissible coarse-graining, and Landauer/Crooks/Jarzynski become precise, calibrated budget statements. New as well is the observable quantity *aging* as the (time-calibrated) accumulation of irreversible *internal* usage, which remains conceptually distinct from proper time and kinematic time dilation.

VIII.1.5 Reading guide

Part VIII is modular: if you only want the decision logic (pass/fail) and the experimental anchors, you can start with the checklist and then jump back selectively to the needed derivations.

Recommended short paths are:

(i) *Classical limit Section* $VIII.3 \rightarrow$ *Section* $VIII.4 \rightarrow$ *Section* $VIII.5$,

(ii) *Thermodynamics/transport Section* $VIII.6 \rightarrow$ *Section* $VIII.7 \rightarrow$ *Section* $VIII.9$,

(iii) *Aging Section* $VIII.8$ (with back-reference to Section VIII.6).

The linear reading order (Section by Section) is:

- **Section VIII.2 - Preliminary foundations & conventions:** Import of the required building blocks and conventions from the FBA – Foundations (notation, units, front/-calibration language), so that the following derivations proceed without silent shifts of meaning.[9]
- **Section VIII.3 - Decoherence, pointer projection & classical limit:** Decoherence as a coarse-graining mechanism; pointer projection as operational classicalization

[5]See FBA Part I: FBA – Foundations, Composition structure; lemma "No budget inflation by rewiring".
[6]See FBA Part VI: Gravity & Geometry from Budget Flows, Introduction and overview.
[7]See FBA Part VII: Constants, Scales & Renormalization, Introduction and overview.
[8]See FBA Part IX: Cosmic Dynamics, Time Dilation & Inflation (TDI), Introduction and overview.
[9]See FBA Part I: FBA – Foundations, Notation, units, front/calibration language.

and derivation of an effective master equation as the first "classical proxy" for later thermodynamic and transport statements.

- **Section VIII.4 - Master → Fokker–Planck → Langevin:** Diffusive scale transition (Kramers–Moyal/Fokker–Planck/Langevin), including clearly stated regularity and truncation assumptions; yields the standard form for rates, drift, and noise terms in the classical regime.
- **Section VIII.5 - Ehrenfest/Hamilton limit & effective trajectories:** Deterministic limiting case (Ehrenfest/Hamilton) as a further reduction of the same coarse-graining structure; explains when "trajectories" are an admissible effective description and where they fail.
- **Section VIII.6 - Thermodynamics I: entropy production, DPI/Spohn & the second law:** Derivation of entropy production and the second law from DPI/Spohn and budget-calibrated balance (with explicit additional assumptions on stationarity/references).
- **Section VIII.7 - Thermodynamics II: fluctuation theorems (Crooks/Jarzynski) in the FBA:** Fluctuation relations as finer tests of the same balance and monotonicity structure; clarifies which assumptions (e.g. time-reversal/detailed-balance proximity, protocol control) are required.
- **Section VIII.8 - Aging in the FBA: definition, metric & observability:** Aging as a calibrated irreversible internal line flow (distinguished from geometric proper time); provides measurable metrics/surrogates and shows which observables can identify aging at all.
- **Section VIII.9 - Nonequilibrium & transport: FDT, Green–Kubo & budget flows:** Connection Section for nonequilibrium: FDT and Green–Kubo as consistency and reconstruction tools, directly tied to the production and balance quantities introduced in Thermodynamics I/II.
- **Section VIII.10 - Comparison & placement relative to the standard picture:** Translation into standard QM/stat-phys language and extraction of what is new in the FBA as a *proxy/regime test* (rather than merely claiming "alternative equations").
- **Section VIII.11 - Summary & checklist (pass/fail):** Compact closure: decision framework, typical failure modes, and a quick smoke-test checklist that can be used directly as an appendix to datasets/simulations.

VIII.2 Preliminary foundations & conventions (Import from Part I: FBA – Foundations)

This Section provides a stable starting point: for the classical limit and thermodynamic statements it is crucial that we do not silently vary (i) admissible microdynamics (CPTP/GKLS), (ii) budget balances, and (iii) the resulting monotonicities. We therefore import the required building blocks unchanged and merely extend the notation where we will later formulate classical generators, drift/diffusion coefficients, and entropy flows precisely.[10]

Imported building blocks (unchanged)

We adopt the following building blocks *without* redefinition from Part I: FBA – Foundations.

- **Sequence of global states & minimal events:** Part I, Chap. I.2 "Global states, frame sequence, and minimal event (ME)"; Part I, Chap./Box I.2 "Co-actuality and refinement invariance".
- **Difference function & operational minimal difference:** Part I, Box I.2 "Difference function & operational minimal difference".
- **Budget calculus (internal/external/irreversible) & balance:** Part I, Chap./Box I.3 "One-step budget & decomposition"; Part I, formula box I.3 "Balance equations"; Part I, lemma I.3 "Refinement invariance of the balance".
- **External calibration & front:** Part I, definition I.3 "Calibration and front costs"; Part I, lemma I.3 "Front bound"; Part I, corollary I.3 "Signal front".
- **Proper time & aging, Minkowski limit:** Part I, definition I.4 "Proper time (proper time)"; Part I, formula box I.4 "Properties of proper time"; Part I, definition I.4 "Aging (irreversible)"; Part I, formula box I.4 "Minkowski limit & quadric"; Part I, lemma I.4 "Time dilation".
- **Admissible dynamics (CPTP/GKLS), DPI/Spohn:** Part I, definition I.5 "Admissible Channels (CPTP)"; Part I, formula I.5 "Kraus/Stinespring"; Part I, lemma I.5 "Measurement as CPTP"; Part I, definition I.5 "GKLS generators (open systems)"; Part I, formula I.5 "Spohn monotonicity"; Part I, lemma I.5 "Semigroup–budget"; Part I, definition/corollary I.5 "DPI arrow & no-recovery".
- **Composition, locality & no-signalling:** Part I, definition I.6 "Symmetric-monoidal structure"; Part I, formula I.6 "Budget additivity"; Part I, lemma I.6 "No-wire inflation & local operations"; Part I, corollary I.6 "Causal cones & local GKLS".

This fixes what counts, in what follows, as already operationally established. What is still missing is a consistent language layer for the transition to effective classical states and for the thermodynamic quantities we will later derive from DPI/Spohn and budget flows. To avoid overloading the import function of this Section, we bundle the conventions into two compact notation boxes (budget/thermo vs. stochastics/GKLS).

[10] See FBA Part I: FBA – Foundations, Chaps. I.2–I.6 (imported building blocks).

Notation & conventions I: budget, time, thermo

- **Discrete vs. continuum:** step index $n \in \mathbb{Z}$; $\delta(\cdot)$ for one-step increments, $d(\cdot)$ for differential quantities; $\sum \delta(\cdot)$ vs. $\int d(\cdot)$.
- **Budget decomposition & time calibration:** $\delta b_{\text{int}} = \delta b_{\text{int}}^{\text{rev}} + \delta b_{\text{irr}}$ with $\delta b_{\text{irr}} \geq 0$ (analogously in the continuum). With κ_τ (and $\alpha_\tau := \kappa_\tau^{-1}$):

$$d\tau_{\text{geo}} = \frac{db_{\text{int}}^{\text{rev}}}{\kappa_\tau}, \quad dA = \frac{db_{\text{irr}}}{\kappa_\tau} \geq 0, \quad d\tau_{\text{tot}} = \frac{db_{\text{int}}}{\kappa_\tau} = d\tau_{\text{geo}} + dA.$$

- **Units/calibration:** c is the calibration constant of the fastest admissible fronts (metrologically fixed); c and k_B remain explicit (no c=1, k_B=1).
- **Thermo conventions:** $Q > 0$ into the system, $W > 0$ work done on the system, $\beta = (k_B T)^{-1}$. For a bath at β: $dS_{\text{env}} = -\beta\, \delta Q$. (Here δQ is thermodynamic notation for an *inexact* differential and must not be confused with $\delta(\cdot)$.)

Notation & conventions II: GKLS, reduction, stochastics

- **GKLS & references:** $\dot{\rho} = \mathcal{L}_t(\rho)$. Common reference: $\mathcal{L}_t(\rho^\star) = 0\ \forall t$. Instantaneous reference: $\mathcal{L}_t(\rho_t^\star) = 0$ (monotonicities relative to $\rho_t^\star$ only with explicitly controlled additional terms). Finite-dimensional by default; in the infinite-dimensional case domains/continuity/smearing must be added explicitly.
- **Coarse–graining:** CPTP reduction $\mathcal{R} : \rho \mapsto p$; effective classical generator K_t for p_t.
- **Fokker–Planck/Langevin (Itô):** $\partial_t p = -\partial_i(a_i p) + \partial_i \partial_j (D_{ij} p)$ with $D \succeq 0$, equivalently $dx = a\, dt + B\, dW_t$ with $2D = BB^\mathsf{T}$ (details in the respective Section).
- **Minimal symbols:** $\|\cdot\|$, $\mathbb{E}[\cdot]$, ln.

VIII.3 Decoherence, pointer projection & classical limit

In the FBA, the classical limit does not arise from additional postulates, but as a robust effective description when suitable *pointer sectors* bundle the relevant degrees of freedom and off-diagonals relax on a short time scale.[1, 2] In this Section we first make the reduction $\rho \mapsto p$ precise, then formulate the associated contractions/monotonicities (DPI/Spohn)[3–5] *under* this reduction, and finally state sufficient conditions under which an (approximately) closed master equation for p_t results.[6–8] [11] [12]

VIII.3.1 Primitives, definitions & reduction maps

We first fix what is meant by a *pointer-stable* decomposition and how classical variables are operationally extracted from ρ. The key point is: the choice of sectors is not mere convention, but is tested against the actual (GKLS) dynamics.[6–8] Only then does "classical" coincide with "stable under admissible dynamics".[1, 2]

Definition VIII.3.1.1: Pointer family & pointer-stable projection

A finite or countable projector family $\{\Pi_x\}_{x\in\mathcal{X}}$ is called a *pointer family* if

1. **Projective resolution:**

$$\Pi_x\Pi_y = \delta_{xy}\Pi_x, \qquad \sum_{x\in\mathcal{X}} \Pi_x = \mathbb{I}.$$

   (If $\mathcal{X}$ is countably infinite, the sum is understood in the usual operator-algebraic sense, e. g. as a strong limit.)

2. **Decoherence stability:** The GKLS dynamics $\dot\rho = \mathcal{L}(\rho)$ damps the off-diagonals $\Pi_x\rho\,\Pi_y$ for $x \neq y$ exponentially on a time scale τ_{dec},[1, 2, 8] i. e. there exist constants $C_{\text{dec}}, \gamma_{\text{dec}} > 0$ such that (for a suitable, dynamics-compatible norm $\|\cdot\|$)

$$\|\Pi_x\rho_t\Pi_y\| \;\leq\; C_{\text{dec}}\, e^{-\gamma_{\text{dec}}t}\,\|\Pi_x\rho_0\Pi_y\|, \qquad (x\neq y), \quad \tau_{\text{dec}} \sim \gamma_{\text{dec}}^{-1}.$$

Given a pointer-stable family, "classical" is initially an *information notion*: we retain the population weights in the sectors and discard coherent cross terms. To ensure this reduction is not merely a snapshot, we additionally fix a canonical re-embedding so that closure errors become explicit.

[11] See FBA Part IV: Dynamics & Measurement (GKLS), Chap. IV.4–IV.6 (decoherence mechanisms, measurement instruments).

[12] See FBA Part III: Quantum Kinematics & CPTP Channels, Chap. III.3–III.5 (CPTP, channels, structures).

Definition VIII.3.1.2: Coarse–graining $\mathcal{R}$ and classical embedding ι

We identify the classical state space with a commutative "register" X with basis $\{|x\rangle\}_{x\in\mathcal{X}}$. The *coarse–graining* is the quantum–to–classical channel[9, 10]

$$\mathcal{R}(\rho) \; := \sum_{x\in\mathcal{X}} \mathrm{tr}(\Pi_x\rho)\,|x\rangle\langle x|, \qquad \text{d. h.} \qquad p_x := \mathrm{tr}(\Pi_x\rho).$$

A canonical *embedding* ι lifts p to the block-diagonal part of the state space, e. g.

$$\iota(p) \; := \sum_x p_x\,\sigma_x,$$

with fixed representatives σ_x on the supports Π_x ($\sigma_x \geq 0$, $\mathrm{tr}\,\sigma_x = 1$, $\Pi_x\sigma_x\Pi_x = \sigma_x$). Then $\mathcal{R}\circ\iota = \mathrm{id}$ on the classical simplex (as states on X).

The central technical question is now: when is the reduced description dynamically closed? This is exactly why we need a precise formulation that separates (i) time-scale separation and (ii) a controlled closure error.

Definition VIII.3.1.3: Sector closure & time-scale separation

We say that $\{\Pi_x\}$ satisfies *sector closure* if there exists a (possibly time-dependent) classical generator K_t (rate matrix) such that for all admissible p

$$\mathcal{R}(\mathcal{L}_t(\iota(p))) = K_t\,p + \mathcal{O}(\varepsilon),$$

where $\varepsilon \ll 1$ denotes a small closure error (e. g. beyond a secular approximation). In addition, let $\tau_{\mathrm{dec}} \ll \tau_{\mathrm{obs}}$ (*time-scale separation*) so that off-diagonals are negligible on observation scales. In the closed limit one understands $\varepsilon \to 0$.

This makes clear what "classical limit" means here: not "ρ becomes classical", but "the projected quantities p form an autonomous stochastic system on the relevant scales". To identify K_t, it is helpful to read off jump rates directly from a Lindblad decomposition.[6–8]

Formula Box VIII.3.1.1: GKLS generator and population rates

Write (time-dependent if driven)[6, 7]

$$\mathcal{L}_t(\rho) = -i[H_t, \rho] + \sum_\alpha \Big(L_{\alpha,t}\rho L_{\alpha,t}^\dagger - \tfrac{1}{2}\{L_{\alpha,t}^\dagger L_{\alpha,t}, \rho\}\Big).$$

We treat p as a column vector and $\dot{p} = K_t p$. For $y \neq x$ define the jump rates

$$K_{yx}(t) \;:=\; \sum_\alpha \mathrm{tr}\Big[\Pi_y\, L_{\alpha,t}\, \sigma_x\, L_{\alpha,t}^\dagger\Big], \qquad K_{xx}(t) := -\sum_{y\neq x} K_{yx}(t).$$

Then K_t is column-stochastic ($\sum_y K_{yx} = 0$) and $K_{yx}(t) \geq 0$ for $y \neq x$.[8]

Hamiltonian part (scope). The term $-i[H_t, \rho]$ contributes to the population balance only if H_t mixes sectors (e. g. $[H_t, \Pi_x] \neq 0$); if $[H_t, \Pi_x] = 0$ or sector-mixing terms are effectively averaged out in a secular limit with $\tau_{\text{dec}} \ll \tau_{\text{obs}}$, then H_t generates primarily phases or drift *within* sectors.[8] In that case, H_t contributes mainly phases or drift *within* the sectors.

This identification is not mere formalism: it makes visible which micro-modes "survive" classically (the rates) and which appear only as sector-internal coordinatization (unitary phases/drift). For details on the GKLS normal form and CPTP structure, it suffices here to refer to the foundations.[13] [14]

VIII.3.2 DPI/Spohn monotonicity under projection

A reduction is physically useful only if it does not violate the basic contractions of admissible dynamics. Intuitively: coarse–graining must not increase distinguishability. This is precisely what yields the classical H-theorem as a "shadow" of the underlying monotonicities.[3–5]

Lemma VIII.3.2.1: Classical H-theorem in the projected Markov flow

Let $p_t := \mathcal{R}(\rho_t)$ and assume sector closure in the sense of Definition VIII.3.1.3, i. e. $\dot{p}_t = K_t p_t + \mathcal{O}(\varepsilon)$. Let $p^\star$ be a positive reference measure that is *jointly* stationary, i. e. $K_t p^\star = 0$ for all t. Then (for finite $\mathcal{X}$, or in the countable case under finite divergence) in the closed limit ($\varepsilon \to 0$)

$$\frac{d}{dt}\, D(p_t \| p^\star) \;\leq\; 0.$$

In addition, pointwise (DPI of the projection)[3, 4]

$$D(p_t \| p^\star) \;=\; D(\mathcal{R}(\rho_t) \| \mathcal{R}(\rho^\star)) \;\leq\; D(\rho_t \| \rho^\star)$$

for any $\rho^\star$ with $\mathcal{R}(\rho^\star) = p^\star$.

[13] See FBA Part I: FBA – Foundations, Chap. I.5 (Kraus/Stinespring, GKLS).

[14] See FBA Part IV: Dynamics & Measurement (GKLS), Chap. IV.2.

Proof Sketch VIII.3.2.1: Classical H-theorem in the projected Markov flow

(1) Classical Markov contraction. For $\dot{p} = K_t p$ and a common stationary $p^\star$, the classical relative entropy $D(p_t \| p^\star)$ is a Lyapunov function; its time derivative can be written as a sum of nonnegative contributions (log-sum / convexity argument), hence $\frac{d}{dt} D(p_t \| p^\star) \leq 0$. For $\dot{p} = K_t p + \mathcal{O}(\varepsilon)$ one correspondingly has $\frac{d}{dt} D(p_t \| p^\star) \leq \mathcal{O}(\varepsilon)$. (If instead of a common $p^\star$ only an instantaneous $p_t^\star$ is available, the familiar additional terms appear; these are used in the thermodynamics part only where they are explicitly controlled.)

(2) DPI of the projection. Since $\mathcal{R}$ is CPTP, for all states ρ, σ: $D(\mathcal{R}(\rho) \| \mathcal{R}(\sigma)) \leq D(\rho \| \sigma)$.[3, 4] Setting $\sigma = \rho^\star$ with $\mathcal{R}(\rho^\star) = p^\star$ yields the pointwise bound.

(3) Link to Spohn (consistency). If $\rho^\star$ is a *common* GKLS reference state ($\mathcal{L}_t(\rho^\star) = 0 \ \forall t$), Spohn additionally yields $\frac{d}{dt} D(\rho_t \| \rho^\star) \leq 0$.[5] Thus the projection does not create an "artificial arrow", but transfers contraction into the classical picture.

This prepares the thermodynamically relevant statement: under admissible projection, irreversibility is not "projected in", but remains visible as a contraction. Together with sector closure, this yields the (approximately) autonomous master dynamics for p_t including Lyapunov structure.

Corollary VIII.3.2.1: Classical master dynamics & contraction

Under sector closure (Definition VIII.3.1.3) p_t satisfies a (time-local) master equation

$$\dot{p}_t = K_t \, p_t + \mathcal{O}(\varepsilon), \qquad \sum_y K_{yx}(t) = 0, \quad K_{yx}(t) \geq 0 \ (y \neq x),$$

and in the closed limit $D(p_t \| p^\star)$ is a Lyapunov function provided $p^\star$ is a common stationary reference measure:

$$\frac{d}{dt} D(p_t \| p^\star) \leq 0 \qquad (\text{or } \leq \mathcal{O}(\varepsilon) \text{ for finite closure error}).$$

The underlying monotonicities (Spohn/DPI) are fixed in the foundations; here we only use them in projected form.[15]

VIII.3.3 Markov closure: sufficient conditions

A master equation is only as good as its closure assumptions. We therefore state a sufficient structural criterion that is often satisfied in applications and justifies the derivation of $\dot{p} = Kp$ in a controlled way.[8]

[15] See FBA Part I: FBA – Foundations, Chap. I.5 (Spohn monotonicity, DPI arrow & no-recovery).

Lemma VIII.3.3.1: Block structure & secular limit ⇒ Markov closure

Let the Lindblad operators $L_{\alpha,t}$ be *jump-selective* with respect to $\{\Pi_x\}$, i. e.

$$L_{\alpha,t} = \sum_{x\to y} \Pi_y L_{\alpha,t} \Pi_x,$$

and let sector-mixing coherent terms be either absent ($\Pi_y H_t \Pi_x = 0$ for $y \neq x$) or effectively averaged out in the secular limit relative to $\tau_{\mathrm{dec}} \ll \tau_{\mathrm{obs}}$.[8] Assume furthermore that σ_x is stationary in each block for the sector-internal Hamiltonian dynamics ($\Pi_x[H_t, \sigma_x]\Pi_x = 0$). Then the population equations close to $\dot{p} = K_t p$ with K_t from Formula Box VIII.3.1.1 (up to $\mathcal{O}(\varepsilon)$).

Proof Sketch VIII.3.3.1: Block structure & secular limit ⇒ Markov closure

Set $\rho = \iota(p) + \delta\rho_{\mathrm{off}}$. The dissipative terms yield (for $y \neq x$) exactly the transitions

$$\dot{p}_y = \sum_{x,\alpha} \mathrm{tr}\Big[\Pi_y L_{\alpha,t} \sigma_x L_{\alpha,t}^{\dagger}\Big]\, p_x + \mathcal{O}(\|\delta\rho_{\mathrm{off}}\|).$$

The secular limit and $\tau_{\mathrm{dec}} \ll \tau_{\mathrm{obs}}$ suppress $\delta\rho_{\mathrm{off}}$.[8] If H_t is block-diagonal (or effectively averaged out), the Hamiltonian part does not contribute to the population balance; within blocks it remains trace-preserving. Positivity and the column-sum property follow from the GKLS structure.

Beyond closure, stationary behavior is crucial because it fixes the reference $p^\star$ for contractions. Detailed balance is not required, but it yields a particularly transparent convergence picture.

Corollary VIII.3.3.1: Stationary measure & detailed balance (sufficient)

Let $\rho^\star$ be a stationary GKLS state ($\mathcal{L}(\rho^\star) = 0$) and assume it is compatible with the embedding, i. e. $\rho^\star$ is block-diagonal in $\{\Pi_x\}$ and can be written as $\rho^\star = \sum_x p_x^\star \sigma_x^\star$ (e. g. with $\sigma_x^\star \propto \Pi_x \rho^\star \Pi_x$). Then $p^\star := \mathcal{R}(\rho^\star)$ is stationary for the projected generator K (in the closed limit).

If moreover for all $x \neq y$ (in a suitable representation) the detailed-balance symmetry holds

$$K_{yx}\, p_x^\star = K_{xy}\, p_y^\star,$$

then *detailed balance* holds. If the projected process is additionally ergodic (e. g. irreducible on the relevant state space), then $D(p_t \| p^\star)$ decreases strictly up to the invariant submodule.

For the classification of stationary states and the role of detailed balance in the GKLS framework we refer to the corresponding treatment.[16]

[16] See FBA Part IV: Dynamics & Measurement (GKLS), Chap. IV.7.

VIII.3.4 Remarks on measurement & consistency

The reduction $\mathcal{R}$ is *not* an additional measurement postulate, but it coincides with Born probabilities provided the POVM elements are the pointer projections.[9–11] What matters is the separation between unselected (contractive) dynamics and selective, instrumented updates.[9, 12]

Born consistency & instruments

For a POVM $\{M_x\}$ with $M_x = \Pi_x$, $\mathcal{R}$ coincides with the Born probability $p_x = \mathrm{tr}(M_x \rho)$.[11] Selective measurements correspond to instrumented CPTP steps,[9, 12] which lie *outside* the unselected GKLS flows; Lemma VIII.3.2.1 and Corollary VIII.3.2.1 always refer to unselected dynamics.

The point is operational: monotonicities (DPI/Spohn) are statements about admissible, unselected processing. Selective updates can seemingly "increase" distinguishability because they introduce conditioning — precisely why we do not treat them here as thermodynamics drivers, but as protocol components (instrument).

VIII.3.5 Positioning & outlook

We have thus established a controlled route $\rho_t \xrightarrow{\mathcal{R}} p_t$, including a closure notion, Markov generator, and an H-theorem in the projected picture. In Section VIII.4 we carry this master dynamics into the continuous scaling limit (Kramers–Moyal/Fokker–Planck/Langevin). In Section VIII.5 we use the reversible part for the Ehrenfest or Hamilton limit. For the Minkowski limit (quadric/Lorentz structure) as the kinematic framework see Part II.[17] These two steps then form the basis for the thermodynamic analysis in Sections VIII.6 and VIII.7.

[17] See FBA Part II: Time, Proper Time & Minkowski Geometry, Chap. II.6 "Budget quadric and Minkowski limit" and Chap. II.7 "Relativity & Lorentz symmetries from the quadric".

VIII.4 Master → Fokker–Planck → Langevin

At the level of discrete sectors, Corollary VIII.3.2.1 yields a master equation. For *continuous* pointer variables (positions, collective coordinates), however, this description is too coarse: many small, fast jumps superpose in such a way that, in the appropriate scaling limit, a smooth convection–diffusion dynamics emerges.[13–15] This transition is crucial, because only then does the bridge to standard statistical physics appear, and the later thermodynamic monotonicities become readable as statements about currents, potentials, and diffusion.

We therefore begin with the rescaling, read off drift and diffusion directly from the jump rates, secure positivity in the limit, and then give the equivalent Langevin trajectory form.[13, 14] Questions of scale and units (in particular $\beta = (k_B T)^{-1}$) are purely calibration questions and are bundled systematically.[18]

VIII.4.1 Scaling assumptions & rescaling

The diffusive limit is a controlled statement about *which* microscopic details disappear under coarse-graining and *which* remain as effective coefficients.[13, 15] We therefore fix explicitly the lattice resolution and (equivalently) the scaling of the rates, so that drift and diffusion later appear as quantities reconstructible unambiguously from the rates.

Definition VIII.4.1.1: Diffusive limit & rescaling

Let $\mathcal{X}_h \subset \mathbb{R}^d$ be a lattice with spacing $h > 0$ and states $x \in \mathcal{X}_h$. For each h, let $p^{(h)}(\cdot, t)$ be given by a master equation $\dot{p}^{(h)} = K_t^{(h)} \, p^{(h)}$.

1. **Small jumps:** Admissible jumps $\delta := y - x$ satisfy $\|\delta\| = \mathcal{O}(h)$.
2. **Diffusive scaling (equivalently as rate scaling):** We consider the macroscopic time t on which the effective motion remains nontrivial. Equivalently, one may introduce a microscopic time s and set $t = h^{-2}s$; then the relevant rates are typically of order h^{-2}, so that the first two (rescaled) jump moments are finite.[13, 15]
3. **Density vs. lattice probability:** $p^{(h)}(\cdot, t)$ is a probability *mass* on $\mathcal{X}_h$. A corresponding density $p(x, t)$ on $\mathbb{R}^d$ is to be understood such that for suitable test functions φ

$$\sum_{x \in \mathcal{X}_h} \varphi(x) \, p^{(h)}(x, t) \;\longrightarrow\; \int_{\mathbb{R}^d} \varphi(x) \, p(x, t) \, dx \qquad (h \to 0),$$

   i. e. informally $p^{(h)}(x, t) \approx p(x, t) \, h^d$ in the smooth regime.
4. **Moment limit:** The rescaled first two Kramers–Moyal coefficients from Formula Box VIII.4.1.1 have finite limiting values $a_i(x, t)$ and $D_{ij}(x, t)$ as $h \to 0$.

Under this scaling, exactly the first two moments of the jump distribution survive in the limit.[13, 15] Thus the effective model is not guessed, but read off from the microdynamics.

[18] See FBA Part VII: Constants, Scales & Renormalization, Chap. VII.1–VII.2 “Calibration & thermal scales”.

Formula Box VIII.4.1.1: Kramers–Moyal coefficients from jump rates

For a lattice $x \in \mathcal{X}_h$ and admissible jumps δ define (for each h)

$$a_i^{(h)}(x,t) \;:=\; \sum_\delta \delta_i\, K^{(h)}_{x+\delta,\,x}(t), \qquad D_{ij}^{(h)}(x,t) \;:=\; \tfrac{1}{2}\sum_\delta \delta_i\,\delta_j\, K^{(h)}_{x+\delta,\,x}(t),$$

where the sums run over all admissible δ (with $\|\delta\| = \mathcal{O}(h)$). In the diffusive limit the limits $a_i = \lim_{h\to 0} a_i^{(h)}$, $D_{ij} = \lim_{h\to 0} D_{ij}^{(h)}$ exist (in an appropriate sense), and $D(x,t) = [D_{ij}(x,t)]$ is symmetric.

VIII.4.2 Fokker–Planck equation & positivity

The central consistency requirement is positivity: the effective equation must not generate negative probabilities.[13, 15] In the FBA this is not an extra postulate, but is secured by the origin of the dynamics: jump rates come from GKLS and are therefore nonnegative; in the diffusive limit this yields a positive semidefinite diffusion matrix.

Lemma VIII.4.2.1: Pawula-type: the diffusive limit yields Fokker–Planck with $D \succeq 0$

Under Definition VIII.4.1.1 and finite second jump moments (and the vanishing of higher-order rescaled Kramers–Moyal terms in the diffusive limit), the limit operator is determined by the coefficients a_i and D_{ij} from Formula Box VIII.4.1.1.[13, 15] *Result:* The density $p(x,t)$ satisfies

$$\partial_t p(x,t) \;=\; -\partial_i\big[a_i(x,t)\,p(x,t)\big] + \partial_i\partial_j\big[D_{ij}(x,t)\,p(x,t)\big],$$

and $D(x,t) = [D_{ij}(x,t)] \succeq 0$. (Well-posedness additionally requires the usual regularity and boundary assumptions; cf. Subsection VIII.4.3.)

Proof Sketch VIII.4.2.1: Pawula-type: the diffusive limit yields Fokker–Planck with $D \succeq 0$

In the diffusive rescaling, jumps of size $\mathcal{O}(h)$ contribute, over macroscopic time $t \sim h^{-2}s$, in a Taylor expansion exactly up to second order to a finite limit operator;[13, 15] the assumption in Definition VIII.4.1.1 ensures that higher rescaled moments vanish in the limit (otherwise a nonlocal or higher-order Kramers–Moyal dynamics remains).

Positivity follows from the structure of the second moments: for any $v \in \mathbb{R}^d$ one already has on the lattice level

$$v^\top D^{(h)}(x,t)\,v = \tfrac{1}{2}\sum_\delta (v\cdot\delta)^2\, K^{(h)}_{x+\delta,x}(t) \;\geq\; 0,$$

since $K^{(h)}_{x+\delta,x}(t) \geq 0$. The limit preserves semidefiniteness, hence $D \succeq 0$, and the limit operator is of (local) Markov diffusion type, which preserves $p \geq 0$.[13]

For thermodynamic evaluation it is helpful to write the Fokker–Planck equation in balance form: then irreversible contributions become visible as currents and can be linked directly to entropy production.[16]

Formula Box VIII.4.2.1: Continuity form & currents

The Fokker–Planck equation is a continuity equation $\partial_t p + \nabla\cdot J = 0$ with current

$$J_i(x,t) \;=\; a_i(x,t)\,p(x,t) - \partial_j\big[D_{ij}(x,t)\,p(x,t)\big].$$

A decomposition $J = J^{\mathrm{rev}} + J^{\mathrm{irr}}$ will later be chosen such that J^{irr} carries the dissipative (budget-irreversible) part and J^{rev} describes the reversible flow components (cf. Sections VIII.6 and VIII.7).

VIII.4.3 Langevin representation (Itô/Stratonovich)

The continuity form describes ensemble evolution. For many applications (simulation, path weights, fluctuation theorems), however, the trajectory description is more basic.[14] It makes explicit which random forces survive in the limit and how they are coupled to diffusion.

Formula Box VIII.4.3.1: Equivalent Langevin equations

Choose $B(x,t)$ such that

$$2\,D(x,t) = B(x,t)\,B(x,t)^{\top}.$$

Then the Itô SDE

$$dx(t) \;=\; a(x(t),t)\,dt \;+\; B(x(t),t)\,dW_t, \qquad \mathbb{E}\big[dW_i(t)\,dW_j(t)\big] = \delta_{ij}\,dt,$$

is equivalent to Lemma VIII.4.2.1.[13, 14]

Itô vs. Stratonovich. Writing the same SDE in Stratonovich form shifts the drift by the usual correction term:[14]

$$a_i^{\circ} \;=\; a_i \;-\; \tfrac{1}{2}\sum_{j,k} B_{jk}\,\partial_{x_j} B_{ik}.$$

Only in special cases can this correction be expressed directly in terms of D (and it then also depends on the choice of a square root B). For example, in 1D with $D = \frac{1}{2}B^2$ the short form $a^{\circ} = a - \frac{1}{2}\,\partial_x D$ holds. In general, the correction is fixed by the concrete micro-limit (time ordering / coarse-graining).

Which interpretation is physically correct is therefore not a mere convention: it is fixed by the micro-limit (time ordering and accumulation limits).[19]

[19] See FBA Part III: Quantum Kinematics & CPTP Channels, Chap. III.5 “Stochastic dilations & time ordering”.

Boundary conditions, manifolds, multiplicative noise

The continuous description requires additional structure as soon as boundaries, curvature, or x-dependent diffusion enter:

1. **Boundary conditions.**
   - *Reflecting:* $n \cdot J = 0$ (no net flux through the boundary).
   - *Absorbing:* $p = 0$ (killing at the boundary; equivalently an incoming flux with no return flux).
2. **Geometry (manifolds).** On a manifold the generator is formulated covariantly; in local charts the balance form remains valid, but divergence/volume element and the concrete realization of diffusion depend on the geometry:

$$\partial_t p + \nabla_i J^i = 0.$$

3. **Multiplicative noise.** For x-dependent $D(x, t)$, the stochastic convention (Itô vs. Stratonovich) is not mere notation:[14] it is fixed by the underlying micro-limit (time ordering / coarse-graining).

VIII.4.4 Detailed balance, stationary measures & FDT

The steps so far yield an effective stochastic dynamics. For thermodynamics one additionally needs a regime statement about *when* this stochastic dynamics admits an equilibrium and *how* deviations from it dissipate.[13, 16] This is exactly where detailed balance enters: it is the condition under which stationary measures take a potential form and linear response is particularly directly linked to fluctuations.

Lemma VIII.4.4.1: Detailed balance ⇒ potential structure

Let $p^\star(x)$ be a positive stationary measure of the Fokker–Planck dynamics, and assume *detailed balance* in the diffusion sense, i. e. the stationary current vanishes:[13]

$$J[p^\star](x) \; = \; a(x)\, p^\star(x) - \partial_j\big(D_{ij}(x)\, p^\star(x)\big) \; \equiv \; 0.$$

Then the drift can be written (pointwise) as

$$a_i(x) \; = \; \partial_j D_{ij}(x) \; + \; D_{ij}(x)\, \partial_j \ln p^\star(x).$$

In particular, in the isothermal equilibrium case $p^\star(x) \propto e^{-\beta\Phi(x)}$ one obtains the gradient form

$$a_i(x) \; = \; \partial_j D_{ij}(x) \; - \; \beta\, D_{ij}(x)\, \partial_j \Phi(x),$$

and for constant D this reduces to $a = -D\,\beta\,\nabla\Phi$.

Corollary VIII.4.4.1: Linear limit & fluctuation–dissipation relation (OU)

Linearizing around a stable fixed point level $x^\star$ and assuming D locally constant,[13, 17]

$$a(x) \approx -\Gamma(x - x^\star), \qquad D \approx \text{const},$$

the stationary covariance $C := \mathbb{E}[(x - x^\star)(x - x^\star)^\top]$ is determined by the Lyapunov equation

$$\Gamma C + C\Gamma^\top = 2D$$

(Ornstein–Uhlenbeck process). In the thermal equilibrium regime with $p^\star \propto e^{-\beta\Phi}$ and a local quadratic approximation $\Phi(x) \approx \frac{1}{2}(x - x^\star)^\top H(x - x^\star)$ one obtains (under the usual Einstein/mobility relation)[16]

$$C = \beta^{-1}H^{-1}, \qquad \Gamma = \mu H, \qquad D = \beta^{-1}\mu$$

for a (symmetric) mobility μ.

This structure is later the technical lever to express entropy production, Landauer bounds, and fluctuation theorems in the same language.[20]

VIII.4.5 Example & check

The following minimal example makes the transition tangible:[13, 14] It shows how a discrete, GKLS-compatible jump dynamics yields, in the scaling limit, a continuous OU dynamics, and how drift, diffusion, and stationary measure fit together.

Birth–death chain ⇒ Ornstein–Uhlenbeck

A one-dimensional chain with jumps $\delta = \pm h$ and (macroscopically scaled) rates

$$K_{x+h,x} = h^{-2}\Big(D + \tfrac{h}{2}a(x)\Big), \qquad K_{x-h,x} = h^{-2}\Big(D - \tfrac{h}{2}a(x)\Big),$$

is well-defined (Markov) for each fixed h if and only if the rates are nonnegative, i. e. $D \pm \frac{h}{2}a(x) \geq 0$ on the considered state range (e. g. for small h and bounded a).[15] In the limit $h \to 0$ it yields (via Formula Box VIII.4.1.1) drift and diffusion

$$a(x) = \lim_{h\to 0} \sum_{\delta=\pm h} \delta\, K_{x+\delta,x}, \qquad D = \lim_{h\to 0} \tfrac{1}{2} \sum_{\delta=\pm h} \delta^2\, K_{x+\delta,x}.$$

Choosing $a(x) = -\Gamma(x - x^\star)$ and constant $D > 0$ gives the OU process

$$dx = -\Gamma(x - x^\star)\, dt + \sqrt{2D}\, dW_t,$$

with stationary Gaussian density $p^\star \propto \exp\big(-\frac{1}{2}(x - x^\star)^2/C\big)$, where C is determined by $2D = 2\Gamma C$ (i. e. $C = D/\Gamma$ in the scalar case).[13, 17] In the thermal equilibrium regime this corresponds to $p^\star \propto e^{-\beta\Phi}$ with quadratic potential Φ and Einstein relation $D = \beta^{-1}\mu$.[16]

[20] See FBA Part X: Predictions, Falsifiability & Bridge FBA → QM ↔ GR, Chap. X.2 "Linear response & test quantities".

VIII.4.6 Positioning

This makes the bridge

$$\dot{p} = Kp \;\longrightarrow\; \partial_t p = \mathcal{L}_{\mathrm{FP}} p \;\longleftrightarrow\; \text{Langevin}$$

explicit: drift and diffusion are identified as limit objects of the jump rates, positivity is secured in the limit, and stationary measures are controllable via detailed balance.[13, 14]

In the next step we extract the reversible budget contribution as a Hamilton/Liouville limit, to understand classical dynamics not only as diffusion, but also as a controlled reversible limit (Section VIII.5). Building on this, Sections VIII.6 and VIII.7 formulate entropy production and fluctuation theorems in exactly this continuous language.

VIII.5 Ehrenfest/Hamilton limit & effective trajectories

Sections VIII.3 and VIII.4 have established the transition $\rho_t \xrightarrow{\mathcal{R}} p_t$ and, further, the continuous limit

$$\partial_t p \;=\; -\partial_i(a_i p) \;+\; \partial_i\partial_j(D_{ij}p)$$

(cf. Corollary VIII.3.2.1, Lemma VIII.4.2.1, and Formula Box VIII.4.2.1).[13, 14] This clarifies *how* a classical ensemble description emerges from GKLS-compatible jump rates.

For the classical limit in the narrower sense, the decisive step is still missing: isolating the *reversible* part that carries the trajectory notion. This is exactly why, in this Section, we separate the drift contribution such that (i) Liouville and Hamilton–Jacobi structures become visible as limiting forms and (ii) effective trajectories appear as a robust mean dynamics of narrow packets. Irreversible corrections (dissipation/diffusion) remain controlled perturbations rather than dominating the picture.[21]

VIII.5.1 Moment equations and the Ehrenfest limit

We begin with moments $\langle f\rangle_t$ of smooth observables $f(x)$ under the Fokker–Planck dynamics from Section VIII.4.[13, 14] This viewpoint is chosen deliberately: it immediately shows which terms transport means (drift) and which merely broaden the distribution (diffusion). This allows the reversible core to be identified precisely as the part that supports a Hamilton structure.

Formula Box VIII.5.1.1: Ehrenfest moment dynamics (convention-consistent)

For smooth f (and sufficiently good decay at infinity / suitable boundary conditions) one has[13, 14]

$$\frac{d}{dt}\langle f\rangle_t = \int f\,\partial_t p\,dx = \underbrace{\langle a\cdot\nabla f\rangle_t}_{\text{drift}} + \underbrace{\langle D:\nabla\nabla f\rangle_t}_{\text{diffusion}}, \qquad D:\nabla\nabla f := D_{ij}\,\partial_i\partial_j f.$$

Specializing to phase-space coordinates $x=(q,\pi)$ and choosing the reversible part as $a^{\mathrm{rev}}(x) = J\nabla H(x)$ (canonical symplectic matrix J), one gets

$$\langle a^{\mathrm{rev}}\cdot\nabla f\rangle_t = \langle\{f,H\}\rangle_t, \qquad \{f,g\} := \nabla f^{\top} J\,\nabla g.$$

This statement is conceptually important: the drift contribution can be parametrized so that it carries a Poisson picture. Thus the Hamilton limit is not "a new postulate", but the limiting form of precisely that drift structure that is read as reversible in the budget picture.

[21] See FBA Part II: Time, Proper Time & Minkowski Geometry, Sec. on the Hamilton limit in the Minkowski reference regime (time, proper time & Minkowski geometry).

Cross reference

The decomposition (reversible/irreversible) is the continuum form of the budget calculus.[a] For the GKLS structure (and the origin of nonnegative rates) see [b]

[a]See FBA Part I: FBA – Foundations, Chap. I.3 "Balance equations".
[b]See FBA Part IV: Dynamics & Measurement (GKLS), Chap. IV.2 "GKLS normal form & positivity".

VIII.5.2 Liouville and Hamilton–Jacobi equation

If one sets the irreversible part to zero in a controlled way ($D \equiv 0$, $a^{\mathrm{irr}} \equiv 0$), a pure transport dynamics remains. This limiting case is the right reference point: only relative to it can one later quantify *when* dissipation is "only a correction" and *when* it is dynamically dominant.

Formula Box VIII.5.2.1: Liouville equation (reversible limit)

Let $x = (q, \pi)$ and $a^{\mathrm{rev}}(x) = J\nabla H(x)$ with canonical symplectic matrix J. Then the density $p(x,t)$ satisfies in the reversible limit

$$\partial_t p(x,t) + \{p, H\}(x,t) = 0, \qquad \{f, g\} := \nabla f^\top J\, \nabla g.$$

The characteristics $t \mapsto x_t$ are canonical trajectories:

$$\dot{x}_t = J\nabla H(x_t).$$

The Liouville form answers the "trajectory question" on the ensemble level: trajectories are the characteristics of the transport operator. To obtain the corresponding "wavefront" / action description, one additionally considers regimes in which p concentrates on a few sheets of a phase function (WKB / large-deviation reading).

Formula Box VIII.5.2.2: Hamilton–Jacobi equation (WKB/LD reading, scope)

In small-noise or large-deviation regimes an eikonal-type representation is typical,

$$p(x,t) \;\propto\; \exp\big(-\lambda\, S(x,t)\big), \qquad \lambda \to \infty,$$

where λ is a (calibrated) large scale parameter (e. g. inverse effective noise strength or inverse packet width). To leading order one obtains a Hamilton–Jacobi structure. In the usual configuration-space form with q one has

$$\partial_t S(q,t) + H(q, \nabla_q S(q,t)) = 0, \qquad \pi = \nabla_q S.$$

Caustics mark the limits of single-sheetedness (multi-valued mapping $q \mapsto \pi$).

Domain of validity & breakdown criteria (operational)

The Liouville/HJ limit is meaningful as a *reversible* approximation limit when two conditions are satisfied simultaneously (here $a = a^{\mathrm{rev}} + a^{\mathrm{irr}}$ is a book-keeping decomposition that is operationalized in the thermodynamics Sections):

1. **Irreversibility is small on observation scales.** On an observation interval Δt, diffusive broadening is small compared to the relevant length scale L,

$$\sqrt{\|D\|\,\Delta t} \ \ll\ L,$$

and the mean displacement induced by a^{irr} is small compared to the reversible drift displacement, evaluated along a representative path $t \mapsto x_t$ (e. g. $x_t = \mu_t$ as mean trajectory),

$$\Big\| \int_{t_0}^{t_0+\Delta t} a^{\mathrm{irr}}(x_t, t)\, dt \Big\| \ \ll\ \Big\| \int_{t_0}^{t_0+\Delta t} a^{\mathrm{rev}}(x_t, t)\, dt \Big\|.$$

2. **The distribution remains narrow on the Ehrenfest scale.** The packet width (e. g. covariance Σ_t) stays small enough that a trajectory/eikonal description is not destroyed by strong dephasing or mixing.

Typical breakdown points.

- *Chaos/mixing:* exponential sensitivity broadens packets rapidly.
- *Caustics/multi-valuedness:* Hamilton–Jacobi becomes multi-valued; wavefronts fold.
- *Strong noise coupling:* D or a^{irr} dominate; trajectories are not robust.

In these regimes, the Langevin/Fokker–Planck methods from Section VIII.4 are decisive.[13, 14]

VIII.5.3 Effective trajectories from narrow packets

The Hamilton formulas provide trajectories as characteristics. For measurement and modelling practice, however, the reverse direction is often decisive: When is it legitimate to extract a "path" from a statistical description $p(x, t)$ without discarding essential information? This becomes legitimate precisely when the distribution stays pointer-narrow and its width grows in a controlled way.[14]

Lemma VIII.5.3.1: Narrow pointer packets ⇒ effective trajectories

Let $p(x,t)$ solve the Fokker–Planck equation from Lemma VIII.4.2.1 with mean μ_t and covariance Σ_t. If $\|\Sigma_t\| \le \delta$ (for small δ) and a and D are locally Lipschitz, then[14]

$$\dot{\mu}_t = a(\mu_t,t) + \mathcal{O}(\|\Sigma_t\|),$$

and

$$\dot{\Sigma}_t = \nabla a(\mu_t,t)\,\Sigma_t + \Sigma_t\,\nabla a(\mu_t,t)^\top + 2\,D(\mu_t,t) + \mathcal{O}(\|\Sigma_t\|^{3/2}).$$

In the reversible case $a = J\nabla H$ and $D = 0$, in particular,

$$\dot{\mu}_t = J\nabla H(\mu_t)$$

up to $\mathcal{O}(\delta)$.

This secures the trajectory notion precisely: the path is not an additional object, but the leading term of the moment dynamics as long as the width remains small.

Proof Sketch VIII.5.3.1: Narrow pointer packets ⇒ effective trajectories

Use Formula Box VIII.5.1.1 and expand $a(x,t)$ and $D(x,t)$ around μ_t (Taylor up to linear order). Then $\mathbb{E}[a(x,t)] = a(\mu_t,t) + \mathcal{O}(\|\Sigma_t\|)$. For the covariance, the standard calculation (equivalently via the Itô form with $BB^\top = 2D$, cf. Formula Box VIII.4.3.1) yields the leading term $\dot{\Sigma} = (\nabla a)\Sigma + \Sigma(\nabla a)^\top + 2D$, plus higher moments controlled for narrow packets by $\|\Sigma_t\|^{3/2}$.[14]

Corollary VIII.5.3.1: Deterministic path as a weak limit

For $\|\Sigma_0\| \to 0$ and $D \equiv 0$, $p(\cdot,t)$ converges weakly to δ_{x_t} with

$$\dot{x}_t = a(x_t,t).$$

For small, smooth D, one obtains a Langevin perturbation around the deterministic path (Formula Box VIII.4.3.1).[14]

VIII.5.4 Example and test case

The linear-quadratic case is more than a toy model: it is a reference test because moments close exactly. This allows one to check directly whether the error control claimed in Lemma VIII.5.3.1 has the correct scaling.

Linear-quadratic case (LQ)

Let $H(x) = \frac{1}{2}x^\top Kx$ with symmetric K. Then $a^{\mathrm{rev}}(x) = JKx$, hence $\nabla a^{\mathrm{rev}} = JK$ is constant, and (for general D)

$$\dot{\mu}_t = JK\mu_t, \qquad \dot{\Sigma}_t = (JK)\Sigma_t + \Sigma_t(JK)^\top + 2D.$$

For $D = 0$ one obtains the closed solution

$$\Sigma_t = M_t\,\Sigma_0\,M_t^\top, \qquad M_t := e^{JKt},$$

i. e. the distribution is transported by the Hamilton flow (Liouville, Formula Box VIII.5.2.1); Σ_t is in general not constant, but follows the linear (symplectic) deformation.

VIII.5.5 Positioning & references

The Hamilton limit realizes the classical trajectory notion as a robust mean-field limit: as long as the distribution remains pointer-narrow, the mean follows a deterministic path, and the width provides the controlled correction. This also makes clear *where* thermodynamics enters: dissipation and diffusion are precisely the terms that broaden trajectories, reduce contrast, and carry entropy production.[16]

The connection to proper-time geometry is worked out in the Minkowski framework.[22] Thermodynamic corrections (dissipation, FDT) are analysed in Sections VIII.6 and VIII.7. For unit and scale questions see [23]

[22]See FBA Part II: Time, Proper Time & Minkowski Geometry, Sec. on the Hamilton limit in the Minkowski reference regime.

[23]See FBA Part VII: Constants, Scales & Renormalization, Chap. VII.1–VII.3 “Calibration, thermal & response scales”.

VIII.6 Thermodynamics I: Entropy production, DPI/Spohn & the second law

In the FBA we formulate the second law directly from GKLS dynamics and the budget calculus. Central are (i) a precise definition of *entropy production* as a quantity of irreversible budget usage, (ii) Spohn/DPI as a monotonicity principle for *unselected* processes, and (iii) the Clausius balance including the Landauer bound in the isothermal limit. [24] [25] [26] The mathematical source is GKLS structure and monotonicity of relative entropy: GKLS [6, 7], Spohn production [5] and data processing [3, 4]. The guiding idea is: *irreversibility* appears as contractive data processing (Spohn/DPI) and is accounted for in the budget picture as a *positive irreversible part.*

VIII.6.1 Definitions & balance equations

We consider a (possibly time-local, CP-divisible) GKLS dynamics $\dot{\rho}_t = \mathcal{L}_t(\rho_t)$ [6–8]. As a thermal reference we use a (stationary or instantaneous) reference state $\rho_t^\star$ with $\mathcal{L}_t(\rho_t^\star) = 0$. [27] [28] External couplings are modelled as *baths* α with inverse temperatures β_α; the associated heat fluxes $\dot{Q}_\alpha$ are counted in the *external* budget.[29] We use throughout the sign convention $\dot{Q}_\alpha > 0$ *into the system.*

[24]See FBA Part I: FBA – Foundations, Sec. I.5 (GKLS, Kraus/Stinespring, Spohn; DPI arrow & no-recovery).

[25]See FBA Part IV: Dynamics & Measurement (GKLS), Secs. IV.2–IV.7 (GKLS generator, stationary states, detailed balance).

[26]See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 (Calibration & thermal scales).

[27]See FBA Part I: FBA – Foundations, Sec. I.5 (GKLS generators; Spohn monotonicity).

[28]See FBA Part IV: Dynamics & Measurement (GKLS), Sec. IV.7 (Stationary states & detailed balance).

[29]See FBA Part I: FBA – Foundations, Sec. I.3 (Balance equations: internal/external/irreversible).

Definition VIII.6.1.1: Entropy production $\dot{\Sigma}$, exchange term Φ (calibration)

Entropies (units). We use $S(\rho) := -\mathrm{tr}(\rho \ln \rho)$ as the *dimensionless* von Neumann entropy (in ln-units, "nats") [18]; the physical entropy is $k_B S$. For logarithms we assume the usual support/full-rank assumptions (on the respective supports); otherwise the relative entropy is not finite and statements are meaningful only after appropriate restriction/regularization.

The *entropy production rate* (unselected) is the Spohn production relative to $\rho_t^\star$ [5]:

$$\dot{\Sigma}(t) \ := \ -\mathrm{tr}\Big(\mathcal{L}_t(\rho_t)\big[\ln \rho_t - \ln \rho_t^\star\big]\Big) \ \geq \ 0.$$

The *integrated* entropy production on $[t_0, t_1]$ is $\Sigma = \int_{t_0}^{t_1} \dot{\Sigma}(t)\, dt$.

The *exchange term* (entropy flow into the environment, in thermal units) is fixed by the balance definition

$$\Phi(t) \ := \ \dot{S}_{\mathrm{sys}}(t) \ - \ \dot{\Sigma}(t)$$

where $\dot{S}_{\mathrm{sys}}(t) := \frac{d}{dt} S(\rho_t)$ is the system entropy rate.

Thermal calibration. In bath models with temperatures β_α^{-1}, Φ is operationally calibrated as

$$\Phi(t) \ = \ \sum_\alpha \beta_\alpha(t)\, \dot{Q}_\alpha(t)$$

where $\dot{Q}_\alpha > 0$ denotes heat absorbed by the system from bath α.

Budget link (thermal account). We read the irreversible internal budget rate in the entropic account as

$$\dot{b}_{\mathrm{irr}}^{(S)}(t) \ := \ k_B\, \dot{\Sigma}(t) \ \geq \ 0, \qquad b_{\mathrm{irr}}^{(S)}[t_0, t_1] = \int_{t_0}^{t_1} k_B\, \dot{\Sigma}(t)\, dt.$$

This form is chosen deliberately so that the monotonicity source is transparent: $\dot{\Sigma}$ is a generator expression that is controlled for unselected GKLS dynamics by Spohn/DPI [3–5]. The exchange term Φ is the (calibrated) bridge to Clausius language.[30]

Formula Box VIII.6.1.1: Clausius balance & system/environment entropies

With the (dimensionless) von Neumann entropy $S(\rho) = -\mathrm{tr}(\rho \ln \rho)$ one has identically

$$\dot{S}_{\mathrm{sys}}(t) \ + \ \dot{S}_{\mathrm{env}}(t) \ = \ \dot{\Sigma}(t), \qquad \dot{S}_{\mathrm{env}}(t) \ := \ -\Phi(t).$$

Under thermal calibration (Definition VIII.6.1.1) this becomes

$$\dot{S}_{\mathrm{env}}(t) \ = \ -\sum_\alpha \beta_\alpha(t)\, \dot{Q}_\alpha(t).$$

In the isothermal one-bath case ($\beta_\alpha \equiv \beta$) this reduces to

$$\dot{S}_{\mathrm{sys}}(t) - \beta\, \dot{Q}(t) = \dot{\Sigma}(t).$$

[30] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 (thermal scales / β-calibration).

VIII.6.2 Spohn monotonicity & the second law

The second law is not a postulate here: it follows as a monotonicity statement from admissibility (GKLS/CP-divisible) and the choice of a suitable reference state $\rho_t^\star$ [5–7]. What matters is the protocol class: we consider *unselected* dynamics (no postselection, no unaccounted feedback step).[31]

Lemma VIII.6.2.1: Spohn monotonicity $\Rightarrow \dot\Sigma \geq 0$ (second law)

Let $\dot\rho_t = \mathcal{L}_t(\rho_t)$ be unselected GKLS/CP-divisible and let $\rho_t^\star$ satisfy $\mathcal{L}_t(\rho_t^\star) = 0$. Then for all t:

$$\dot\Sigma(t) \;\geq\; 0.$$

Under thermal calibration (Definition VIII.6.1.1), Formula Box VIII.6.1.1 implies the Clausius inequality

$$\dot S_{\mathrm{sys}}(t) \;\geq\; \sum_\alpha \beta_\alpha(t)\, \dot Q_\alpha(t).$$

In the special case $\rho_t^\star \equiv \rho^\star$ (time-independent) this is equivalently

$$\dot\Sigma(t) \;=\; -\frac{d}{dt} D(\rho_t \| \rho^\star) \;\geq\; 0,$$

consistent with data processing of relative entropy [3, 4].

Corollary VIII.6.2.1: Integrated second law

For any interval $[t_0, t_1]$ one has

$$\Sigma[t_0, t_1] \;=\; \int_{t_0}^{t_1} \dot\Sigma(t)\, dt \;\geq\; 0.$$

In particular, Formula Box VIII.6.1.1 yields the integrated Clausius form

$$\Delta S_{\mathrm{sys}}[t_0, t_1] \;-\; \int_{t_0}^{t_1} \sum_\alpha \beta_\alpha(t)\, \dot Q_\alpha(t)\, dt \;\geq\; 0.$$

In the isothermal one-bath case: $\Delta S_{\mathrm{sys}} - \beta\, Q \geq 0$ with $Q = \int \dot Q\, dt$ and $Q > 0$ into the system.

VIII.6.3 Landauer bound in the FBA

In the FBA, information erasure is a special case: it enforces a contraction (reset/coarse-graining) and therefore requires positive irreversible internal budget. In the isothermal limit this minimal-cost statement becomes the Landauer bound [19, 20] and thus a very direct pass/fail test for the budget interpretation of "information".

[31] See FBA Part III: Quantum Kinematics & CPTP Channels, Secs. III.4–III.5 (Instruments & selective operations).

Corollary VIII.6.3.1: Landauer bound (isothermal, unselected)

Consider an isothermal one-bath protocol at temperature T with $\beta = (k_B T)^{-1}$ and sign convention $Q > 0$ *into the system*. If a CPTP/GKLS protocol enforces a reduction of the system entropy by

$$\Delta S^{\downarrow}_{\rm sys} \;:=\; S(\rho_{t_0}) - S(\rho_{t_1}) \;\geq\; 0,$$

then the heat dumped to the bath

$$Q_{\rm bath} \;:=\; -\int_{t_0}^{t_1} \dot{Q}(t)\, dt \;\geq\; 0$$

is bounded from below by

$$Q_{\rm bath} \;\geq\; k_B T\, \Delta S^{\downarrow}_{\rm sys}.$$

Equivalently: $\beta Q_{\rm bath} \geq \Delta S^{\downarrow}_{\rm sys}$. If the erasure is specified as deletion of ΔI *bits*, then $\Delta S^{\downarrow}_{\rm sys} = \Delta I \,\ln 2$. Equality is achievable only in the quasistatic (reversible) limit under suitable equilibrium/balance conditions.

VIII.6.4 Additivity, locality & measurements

For composite systems, $\dot{\Sigma}$ must be compatible with the composition structure. At the same time it is essential to cleanly separate measurement and feedback protocols from the unselected GKLS flow: only in the unselected picture do the monotonicities constitute "hard" second-law statements. [32] [33]

Lemma VIII.6.4.1: Composition, entropy flows & correlation term

Let a composite system AB be locally coupled to independent baths such that heat flows add ($\Phi_{AB} = \Phi_A + \Phi_B$) and the GKLS generator contains no interaction terms ($\mathcal{L}_{AB} = \mathcal{L}_A \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_B$). Then identically

$$\dot{\Sigma}_{AB}(t) \;=\; \dot{\Sigma}_A(t) + \dot{\Sigma}_B(t) \;-\; \frac{d}{dt} I(A\!:\!B)_t, \qquad I(A\!:\!B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

In particular, $\dot{\Sigma}_{AB} = \dot{\Sigma}_A + \dot{\Sigma}_B$ if $I(A:B)_t$ is constant/negligible on the considered protocol class (e. g. for strictly decoupled dynamics that preserves product states).

Selectivity, feedback & limits of monotonicity

Selective (postselected) protocols are *not* covered by Lemma VIII.6.2.1; along individual branches $\dot{\Sigma}$ can effectively appear "negative" because one considers conditioning/information extraction without the unselected average. Averaged over outcomes (unselected), monotonicity is restored.

[32] See FBA Part I: FBA – Foundations, Sec. I.6 (Composition, locality & no-signalling).

[33] See FBA Part III: Quantum Kinematics & CPTP Channels, Secs. III.4–III.5 (Instruments; selective vs. unselected).

VIII.6.5 Example: two-level system in thermal contact

In the minimal model the structure becomes particularly transparent: detailed balance fixes the stationary measure, and $\dot{\Sigma} \geq 0$ becomes an explicit sum form over transitions [8, 14].

Qubit with a thermal bath (detailed balance)

Setup. Let $H = \frac{\hbar\omega}{2}\sigma_z$ and a thermal GKLS dissipator with excitation/relaxation rates

$$\gamma_\downarrow = \gamma\,(n_\beta + 1), \qquad \gamma_\uparrow = \gamma\, n_\beta, \qquad n_\beta := \left(e^{\beta\hbar\omega} - 1\right)^{-1}.$$

Then $\gamma_\uparrow/\gamma_\downarrow = e^{-\beta\hbar\omega}$ (local detailed balance).

Population dynamics (classical two-state flow). For $p_e(t) = \langle e|\rho_t|e\rangle$, $p_g(t) = 1 - p_e(t)$ one has

$$\dot{p}_e = -\gamma_\downarrow p_e + \gamma_\uparrow p_g, \qquad \dot{p}_g = -\dot{p}_e.$$

The stationary state satisfies

$$\frac{p_e^\star}{p_g^\star} = \frac{\gamma_\uparrow}{\gamma_\downarrow} = e^{-\beta\hbar\omega}.$$

Entropy production. With the (net) transition current $J := \gamma_\downarrow p_e - \gamma_\uparrow p_g$, the (total) EP rate is

$$\dot{\Sigma} = J\,\ln\frac{\gamma_\downarrow p_e}{\gamma_\uparrow p_g} = J\,\ln\frac{p_e/p_g}{p_e^\star/p_g^\star} \;\geq\; 0,$$

since $(x-y)\ln(x/y) \geq 0$ for $x = \gamma_\downarrow p_e$, $y = \gamma_\uparrow p_g$. This is the explicit two-level instance of Lemma VIII.6.2.1.

VIII.6.6 Positioning & outlook

We have established $\dot{\Sigma}$ as a precise quantity of irreversible budget usage and deduced the second law from Spohn/DPI. Section VIII.7 extends this to *fluctuation theorems* (Crooks/Jarzynski) [21, 22] and connects budget flows to path probabilities. For experimental tests (linear response/FDT) see also Section VIII.9 and the test playbook.[34]

[34] See FBA Part X: Predictions, Falsifiability & Bridge FBA $\to$ QM $\leftrightarrow$ GR, Sec. X.2 (Test quantities & protocols).

VIII.7 Thermodynamics II: Fluctuation theorems (Crooks/-Jarzynski) in the FBA

Building on Section VIII.6 (entropy production & the second law), we formulate fluctuation theorems for *finite* protocols in the isothermal limit [16, 21–23]. The starting point are the effective classical descriptions from Sections VIII.3 and VIII.4 as well as the concrete continuum forms from Corollary VIII.3.2.1, Lemma VIII.4.2.1, and Formula Box VIII.4.2.1. The central statements are (i) an *integral* fluctuation theorem for the total entropy production, (ii) the *Crooks* relation, and (iii) the *Jarzynski* equality. [35] [36]

VIII.7.1 Setting: protocols, paths, time reversal

We consider a time-dependent coupling/control $\lambda_t \in \Lambda$ on the effective space of continuous pointer variables x. The dynamics is given by a master or Fokker–Planck equation; paths $\gamma = \{x_t\}_{t\in[0,\tau]}$ carry budget flows (work W, heat Q) according to the current representation in Formula Box VIII.4.2.1 as well as the budget balance from the FBA – Foundations. [37] Throughout we use the sign conventions fixed in Section VIII.6: $Q > 0$ means heat *into the system* and $W > 0$ work *done on the system.* This makes irreversibility directly measurable at the trajectory level: as a ratio of forward and backward path measures [16, 23].

Definition VIII.7.1.1: Protocol, path measure & microreversibility

Protocol. A piecewise smooth control $\lambda : [0, \tau] \to \Lambda$. Forward: $\lambda_F(t) = \lambda_t$. Backward: $\lambda_R(t) = \lambda_{\tau-t}$.

Paths & path measures. Path $\gamma = \{x_t\}_{t\in[0,\tau]}$. Forward start (isothermal equilibrium)

$$p_0^{\mathrm{eq}}(x) = Z_0^{-1} e^{-\beta E(x,\lambda_0)}, \qquad Z_0 := \int e^{-\beta E(x,\lambda_0)} dx,$$

path measure $\mathbb{P}_F[\gamma]$. Backward start

$$p_\tau^{\mathrm{eq}}(x) = Z_\tau^{-1} e^{-\beta E(x,\lambda_\tau)}, \qquad Z_\tau := \int e^{-\beta E(x,\lambda_\tau)} dx,$$

time reversal $\tilde{\gamma}_t := \Theta\, \gamma_{\tau-t}$ with parity operator Θ (e. g. $\Theta(q, p) = (q, -p)$), path measure $\mathbb{P}_R[\tilde{\gamma}]$.

Local detailed balance (ldb). For elementary transitions (jump or diffusion step) one has [16, 23]

$$\ln \frac{w_F(x' \mid x; \lambda_t)}{w_R(\Theta x \mid \Theta x'; \lambda_{\tau-t})} = -\,\beta\, q_t(x \to x')\,,$$

where $w_{F/R}$ are the respective transition rates (discrete) or short-time propagator densities (diffusion). Here q_t denotes the *instantaneous* heat absorbed by the *system* (convention: positive “into the system”).

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[35] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 (Calibration & thermal scales).

[36] See FBA Part IV: Dynamics & Measurement (GKLS), Sec. IV.7 (Stationary states & detailed balance).

[37] See FBA Part I: FBA – Foundations, Sec. I.3 (Balance equations).

VIII.7.2 Integral fluctuation theorem (IFT)

At the path level, the total entropy production $\Sigma[\gamma]$ measures the irreversibility of the trajectory; formally, Σ is the log-density of a Radon–Nikodym ratio of the path measures [16, 23].

Lemma VIII.7.2.1: IFT for the total entropy production

Let

$$\Sigma[\gamma] \;:=\; \ln \frac{\mathbb{P}_F[\gamma]}{\mathbb{P}_R[\tilde{\gamma}]}\,.$$

Then

$$\left\langle e^{-\Sigma[\gamma]} \right\rangle_F \;=\; 1\,, \qquad \text{hence} \qquad \langle \Sigma \rangle_F \;\geq\; 0.$$

Formula Box VIII.7.2.1: Trajectory EP and EP–work relation (isothermal)

For an isothermal bath ($\beta = \text{const.}$) and the sign convention $Q > 0$ "into the system", the total trajectory EP can be written as

$$\Sigma[\gamma] \;=\; \Delta s_{\text{sys}}[\gamma] \;-\; \beta\, Q[\gamma], \qquad \Delta s_{\text{sys}}[\gamma] := -\ln p(x_\tau, \tau) + \ln p(x_0, 0),$$

[16, 23]. Under the equilibrium boundary conditions of Definition VIII.7.1.1 this reduces to

$$\Sigma[\gamma] \;=\; \beta(W[\gamma] - \Delta F), \qquad F(\lambda) := -\beta^{-1} \ln Z(\lambda), \quad \Delta F := F(\lambda_\tau) - F(\lambda_0),$$

and $W_{\text{diss}} := W - \Delta F$ is the dissipated work.

VIII.7.3 Crooks relation

Crooks is the "differentiated" version of the IFT and relates the *work distributions* of the forward and reverse processes [21].

Formula Box VIII.7.3.1: Crooks fluctuation relation

Under isothermal equilibrium starts,

$$\frac{P_F(W)}{P_R(-W)} \;=\; \exp[\beta\,(W - \Delta F)].$$

The crossing point $P_F(W) = P_R(-W)$ lies at $W = \Delta F$.

VIII.7.4 Jarzynski equality & the second law

Jarzynski follows as a moment identity from Crooks/IFT and makes ΔF experimentally accessible [21, 22].

Formula Box VIII.7.4.1: Jarzynski equality

$$\left\langle e^{-\beta W} \right\rangle_F = e^{-\beta \Delta F}.$$

Corollary VIII.7.4.1: Second law from Jarzynski (Jensen)

From Formula Box VIII.7.4.1 it follows that $\langle W \rangle_F \geq \Delta F$ and hence

$$\beta \langle W_{\mathrm{diss}} \rangle_F = \langle \Sigma \rangle_F \geq 0.$$

VIII.7.5 Generalizations & practical aspects

For applications, nonequilibrium stationarity, multiple baths, and estimation procedures are particularly important [16].

NESS/Hatano–Sasa, multiple baths, measurement practice

1. **Nonequilibrium steady state (NESS) / Hatano–Sasa.** For a stationary nonequilibrium measure $p^{\mathrm{ss}}(x|\lambda)$ and $\phi(x,\lambda) := -\ln p^{\mathrm{ss}}(x|\lambda)$ one has the Hatano–Sasa IFT [24]:

$$Y[\gamma] := \int_0^\tau \dot{\lambda}_t \cdot \partial_\lambda \phi(x_t, \lambda_t)\, dt, \qquad \langle e^{-Y[\gamma]} \rangle = 1.$$

   (In models with local detailed balance, Y can be identified with the "excess" part of the dissipation.)

2. **Multiple baths.** With several baths α at inverse temperatures β_α, one generally has [16]

$$\Sigma[\gamma] = \Delta s_{\mathrm{sys}}[\gamma] - \sum_\alpha \beta_\alpha Q_\alpha[\gamma],$$

   with the convention $Q_\alpha > 0$ "into the system".

3. **ΔF estimation from nonequilibrium work.** Jarzynski is exponential reweighting; in practice forward/reverse combination estimators such as Bennett acceptance ratio are central [25].

4. **Experiment / data analysis.** *Work* from trajectory work (force–distance/parametric work), *heat* from current data (cf. Formula Box VIII.4.2.1) and power flows.

VIII.7.6 Positioning & outlook

The fluctuation theorems sharpen the second law at the path level and identify Σ with the *irreversible internal* budget along individual trajectories [16, 23]. In Section VIII.8 we define and operationalize *aging* as the integrated irreversible internal budget flow along worldlines and sharply separate it from proper time; observation and comparison protocols are given explicitly.

VIII.8 Aging in the FBA: Definition, Metric & Observability

In this Section we make *aging* precise as the integrated *irreversible internal* budget usage along a worldline, and we distinguish it from proper time. The point is not semantic, but operational: $\tau_{\rm geo}$ measures the kinematical–geometric (reversible) component, whereas A accumulates exactly the component generated by irreversible internal dynamics (dissipation/information loss).

Two bridges make A observable: (i) the link to entropy production from Spohn/DPI (Lemma VIII.6.2.1) and (ii) the classical effective dynamics from Sections VIII.3 and VIII.4, from which practical estimators follow in master- and Fokker–Planck language (Corollary VIII.3.2.1, Lemma VIII.4.2.1, and Formula Box VIII.4.2.1).[38][39]

VIII.8.1 Definition & basic structure

We begin with the definition along a worldline γ. What matters is: in the FBA, A is a *calibrated time quantity* that arises from irreversible *internal* budget usage. For clarity we separate (i) the *budget accumulation* and (ii) the *aging time* derived from it via the time calibration. (The parametrization along γ is arbitrary; A is parametrization-invariant as a line functional.)

Definition VIII.8.1.1: Aging (operational quantity)

Let γ be a worldline of a system in the Frame–Budget Approach, and let $\dot{b}_{\rm irr,int}(\lambda) \geq 0$ denote the *irreversible internal* budget rate in an arbitrary step-/continuum parameter λ. The *irreversible internal budget accumulation* is

$$B_{\rm irr,int}[\gamma] \;:=\; \int_\gamma \dot{b}_{\rm irr,int}(\lambda)\, d\lambda \;\geq 0.$$

With the time calibration κ_τ (so that $d\tau = db/\kappa_\tau$) we define *aging* as

$$A[\gamma] \;:=\; \frac{1}{\kappa_\tau}\, B_{\rm irr,int}[\gamma] \;=\; \frac{1}{\kappa_\tau} \int_\gamma \dot{b}_{\rm irr,int}(\lambda)\, d\lambda \;\geq 0.$$

Proper time plays a double role in the FBA: as the geometric–kinematical component in the Minkowski limit and as the total, budget-side accumulation including irreversible contributions. The following decomposition is the formal dividing line between "kinematics" and "thermo-dynamics" along the same worldline.

[38] See FBA Part II: Time, Proper Time & Minkowski Geometry, Sec. II.4 and Secs. II.6–II.7 (Proper time & aging; Minkowski limit & Lorentz kinematics).

[39] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 (thermal scales).

Formula Box VIII.8.1.1: Proper-time decomposition: geometric vs. aging

The *total* proper time along γ decomposes as

$$\tau_{\text{tot}}[\gamma] \;=\; \tau_{\text{geo}}[\gamma] \;+\; A[\gamma],$$

where τ_{geo} stems from the *reversible internal* budget (Minkowski limit) and A accumulates exclusively the *irreversible internal* part ([a]).

[a] See FBA Part II: Time, Proper Time & Minkowski Geometry, Sec. II.4 (Proper time & aging).

This also makes clear which statements do *not* follow automatically: A difference in τ_{geo} (time dilation, kinematics) is, by itself, not a difference in A, as long as the locally relevant irreversible internal dynamics is realized unchanged. Conversely, A can vary without changing τ_{geo}.

Lemma VIII.8.1.1: Basic properties of A

For arbitrary concatenated segments $\Gamma = \Gamma_1 \circ \Gamma_2$ and reparametrized $\tilde{\gamma}$, the following hold:

1. **Nonnegativity:** $A[\gamma] \geq 0$.
2. **Additivity:** $A[\Gamma] = A[\Gamma_1] + A[\Gamma_2]$.
3. **Parametrization invariance:** $A[\tilde{\gamma}] = A[\gamma]$.
4. **Decoupling from kinematics (Scope):** If the same internal irreversible dynamics is realized *as a function of the same local process parametrization* (e. g. as a function of local proper time, or under identical internal couplings), then a purely kinematical change (time dilation) changes τ_{geo}, but not the functional A.

VIII.8.2 Thermal calibration & entropy production

To make A measurable, we couple it to entropy and heat flows. The core mechanism is the same as in Section VIII.6: Spohn/DPI yields $\dot{\Sigma} \geq 0$ as an irreversibility measure for *unselected* flows.[5–7] In the FBA, bookkeeping is added: contributions modeled as bath-/exchange couplings are explicitly booked in the *external* budget account; the remaining, model-dependently assigned residual is *internally* booked irreversibility.

Definition VIII.8.2.1: "Internal" vs. "external" in Σ (bookkeeping decision)

Write the (effective) unselected dynamics as a sum of bookkeeping-assigned contributions, e. g. on the GKLS level

$$\mathcal{L}_t \;=\; \mathcal{L}_t^{\mathrm{rev}} \;+\; \sum_{\alpha\in\mathrm{ext}} \mathcal{D}_t^{(\alpha)} \;+\; \sum_{\ell\in\mathrm{int}} \mathcal{D}_t^{(\ell)},$$

where α denotes "external" (bath-/exchange account) and ℓ denotes "internal" (internal dissipation/loss channels in the model account). Then $\dot{\Sigma}(t)$ is the total EP from Definition VIII.6.1.1, and $\dot{\Sigma}_{\mathrm{int}}(t)$ denotes the EP carried by the dissipative terms classified as "internal" (operationally: the same EP formula, but with the sum/integration taken only over the internally booked channels).

In the isothermal single-bath case, the conversion is particularly transparent: $\beta^{-1} = k_B T$ sets the thermal energy scale with which dimensionless entropy production can be translated into an energy equivalent; with κ_τ this becomes an aging time.

Formula Box VIII.8.2.1: Calibration via entropy production (isothermal)

For coupling to a bath with constant $\beta = (k_B T)^{-1}$ and unselected dynamics, define $\dot{\Sigma}_{\mathrm{int}}(t) \geq 0$ as the *internally* booked part of the entropy production (in the sense of Definition VIII.8.2.1). Then the operative calibration/estimation formula is

$$\dot{B}_{\mathrm{irr,int}}(t) \;=\; \beta^{-1}\,\dot{\Sigma}_{\mathrm{int}}(t), \qquad \dot{A}(t) \;=\; \frac{1}{\kappa_\tau}\,\dot{B}_{\mathrm{irr,int}}(t) \;=\; \frac{\beta^{-1}}{\kappa_\tau}\,\dot{\Sigma}_{\mathrm{int}}(t).$$

Integrating over $[t_0, t_1]$ yields

$$A[t_0, t_1] \;=\; \frac{\beta^{-1}}{\kappa_\tau}\int_{t_0}^{t_1} \dot{\Sigma}_{\mathrm{int}}(t)\,dt.$$

Interpretation: β^{-1} sets the thermal scale (energy equivalent per unit EP), κ_τ sets the time calibration (time per budget unit).

Cross-reference

$\dot{\Sigma} \geq 0$ from Lemma VIII.6.2.1; thermal balance in the isothermal limit from Formula Box VIII.6.1.1. The internal/external assignment is performed via the budget calculus of the FBA – Foundations (Sec. I.3); cf. Definition VIII.8.2.1. For path interpretations (IFT/Crooks/Jarzynski) cf. Section VIII.7.

Practical remark: "internal" here does not mean "inside the device" in a colloquial sense, but "in the internal budget account" of the model. Which dissipation channels end up there is an *explicit* modeling and bookkeeping decision; this is precisely why the separation is testable in the FBA and not mere convention.

VIII.8.3 Operational estimators (Master/FP)

The classical generators and current forms established in Sections VIII.3 and VIII.4 immediately yield practical estimators for $\dot{A}$. We give both standard cases: (i) discrete Markov jumps (master equation) and (ii) diffusions (FP/Langevin). In both cases the structural core is the same: entropy production is a functional of irreversible currents (log-ratio or quadratic form).[16, 26, 27]

Formula Box VIII.8.3.1: Markov jumps: EP density & A

For a time-local master dynamics $\dot{p} = K_t p$ with currents $J_{yx} = K_{yx}p_x - K_{xy}p_y$, the (classical) entropy production rate is

$$\dot{\Sigma}(t) = \tfrac{1}{2} \sum_{x,y} J_{yx}(t) \ln \frac{K_{yx}(t)\, p_x(t)}{K_{xy}(t)\, p_y(t)} \geq 0.$$

The *internal* part $\dot{\Sigma}_{\text{int}}$ (sum only over internally booked transitions) yields, via Formula Box VIII.8.2.1,

$$\dot{A}(t) = \frac{\beta^{-1}}{\kappa_\tau} \dot{\Sigma}_{\text{int}}(t).$$

For continuous variables, the FP current form is particularly useful because it expresses EP as "dissipation per density".[13, 16]

Formula Box VIII.8.3.2: Fokker–Planck: current formula & A

For $\partial_t p = -\nabla \cdot J$ with current J from Formula Box VIII.4.2.1 and a decomposition $J = J^{\text{rev}} + J^{\text{irr}}$, one has

$$\dot{\Sigma}(t) = \int \frac{\left(J^{\text{irr}}(x,t)\right)^\top D(x,t)^+ \, J^{\text{irr}}(x,t)}{p(x,t)} \, dx \geq 0,$$

where D^+ denotes the (Moore–Penrose) pseudoinverse of D (in the full-rank case $D^+ = D^{-1}$).[a] The internally booked part $\dot{\Sigma}_{\text{int}}$ (according to a model/bookkeeping assignment) then enters

$$\dot{A}(t) = \frac{\beta^{-1}}{\kappa_\tau} \dot{\Sigma}_{\text{int}}(t)$$

accordingly.

[a] This is the robust version if D is only semidefinite; effectively, dissipation occurs only along the diffusion-driven directions.

VIII.8.4 Operational separation: aging vs. proper time

The conceptual separation becomes experimentally sharp when "same τ_{geo}" and "same A" can be varied independently. The following statement provides exactly these two control regimes.

Lemma VIII.8.4.1: Same proper time, different aging (and vice versa)

Consider two identical systems S_1, S_2.

(i) Same τ_{geo}, variable A. If both systems follow the same worldline (hence the same τ_{geo}) and their internally dissipative couplings differ ($\mathcal{L}^{(1)}_{\text{irr,int}} \neq \mathcal{L}^{(2)}_{\text{irr,int}}$), then in general $A_1 \neq A_2$.

(ii) Different kinematics, same A (Scope). If the internally dissipative couplings are realized identically *in the same local process parametrization* (e. g. identical internal GKLS coupling as a function of local proper time), then one can have $A_1 = A_2$ despite $\tau_{\text{geo},1} \neq \tau_{\text{geo},2}$.

Proof Sketch VIII.8.4.1: Same proper time, different aging (and vice versa)

(i) From Formula Box VIII.8.2.1 we have $\dot{A} \propto \dot{\Sigma}_{\text{int}}$. Different internal dissipative terms change $\dot{\Sigma}_{\text{int}}$ and thus the integral A, independent of whether τ_{geo} is the same.

(ii) If the internal irreversible dynamics is realized identically as a function of the same local parametrization (e. g. as a function of τ_{geo}), then $\dot{\Sigma}_{\text{int}}$ is the same in that parametrization; hence A is the same, while a purely kinematical change modifies τ_{geo}.

VIII.8.5 Measurement protocols & observability

We sketch two minimally invasive protocols that make A experimentally accessible. Both are designed so that only *one* of the two quantities (τ_{geo} or A) is varied deliberately, while the other serves as a control variable.

Comparison clocks with controlled dissipation

A1: Same τ_{geo}, variable A. Two identical clocks U_{cold} and U_{hot} move along the same worldline (same τ_{geo}). An additional local noise source (internal GKLS operator L_ℓ with rate κ) drives only U_{hot} dissipatively. From the measured internal entropy production (or equivalently from calorimetrically determined internal dissipation in the isothermal limit) one obtains

$$\Delta A = \frac{\beta^{-1}}{\kappa_T} \int_{t_0}^{t_1} \left(\dot{\Sigma}^{\text{hot}}_{\text{int}}(t) - \dot{\Sigma}^{\text{cold}}_{\text{int}}(t) \right) dt.$$

A2: Different kinematics, same A (Scope). Two identical clocks with identical internal GKLS couplings are sent on two worldlines with different time dilation. Calorimetric measurement of the internally booked dissipation, or equivalently Σ_{int}, yields $A_1 \approx A_2$, although $\tau_{\text{geo},1} \neq \tau_{\text{geo},2}$, *provided* the internal coupling is realized in the same local parametrization (e. g. as a function of local proper time).

VIII.8.6 Pass/fail indicators (minimal checklist)

To conclude, we collect observable criteria that test the separation between τ_{geo} and A. These points are phrased deliberately so that they can be used as data checks.

Core indicators for aging

- **Nonnegativity/monotonicity:** $\dot{A} \geq 0$ (from $\dot{\Sigma}_{\text{int}} \geq 0$ via Formula Box VIII.8.2.1).
- **Additivity:** $A[\Gamma_1 \circ \Gamma_2] = A[\Gamma_1] + A[\Gamma_2]$.
- **Decoupling from time dilation (Scope):** Varying kinematics while keeping the internal GKLS coupling constant in the local parametrization changes τ_{geo}, not A.
- **Landauer consistency (isothermal):** If a reset enforces an information reduction ΔI, then Corollary VIII.6.3.1 implies for the internally irreversible aging time
$$A \;\geq\; \frac{\beta^{-1}}{\kappa_\tau}\,\Delta I.$$
[19, 20]

VIII.8.7 Classification & outlook

Aging is thus a *thermodynamic-kinetic* quantity of the FBA, separated from pure kinematics (time dilation) and directly estimable via entropy/heat flows. Section VIII.9 uses this separation for transport and response theory (FDT, Green–Kubo) out of equilibrium; Section VIII.10 discusses experimental tests and falsifiability as well as bridges to standard approaches.

VIII.9 Nonequilibrium & Transport: FDT, Green–Kubo & budget flows

This Section connects the FBA budget picture to linear response theory.[28, 29] Starting from the continuous description (Sections VIII.3 and VIII.4; Corollary VIII.3.2.1, Lemma VIII.4.2.1, and Formula Box VIII.4.2.1) and entropy production (Section VIII.6; Lemma VIII.6.2.1 and Formula Box VIII.6.1.1), we formulate (i) force–flux pairs, (ii) fluctuation–dissipation relations (FDT), and (iii) Green–Kubo relations together with Onsager–Casimir symmetries. The results yield positivity and symmetry statements for transport coefficients, directly interpretable as properties of *irreversible* budget usage. [40] The microreversible equilibrium assumptions (detailed balance/ldb) are discussed in [41] as well as in the path formulation of Section VIII.7 (cf. Definition VIII.7.1.1).

VIII.9.1 Fluxes, forces & entropy production (linear limit)

We linearize around a stationary measure $p^\star$ (in the FDT/Onsager part: equilibrium with microreversibility/ldb). Irreversible currents stem from J^{irr} in the Fokker–Planck form (Formula Box VIII.4.2.1) or from jump currents J_{yx} (master equation). The purpose of this linearization is twofold: (i) it provides a canonical definition of transport coefficients as responses to small drivings, and (ii) it turns entropy production into a quadratic form whose positivity immediately yields bounds on $L_{\alpha\beta}$.

Definition VIII.9.1.1: Thermodynamic forces & fluxes

Let $\{J_\alpha(t)\}$ be observable *irreversible* fluxes (e. g. heat, particle, or momentum currents), and let $\{f_\alpha(t)\}$ be the associated *affine forces* (gradients of $\beta\mu$, βu, $\beta\boldsymbol{v}$, etc.). In the (nearly) stationary regime we interpret $J_\alpha(t)$ as (macroscopic) fluxes or expectation values, so that in the linear regime

$$\dot{\Sigma}(t) \;=\; \sum_\alpha f_\alpha(t)\, J_\alpha(t) \;\geq 0,$$

consistent with the Clausius form Formula Box VIII.6.1.1 (isothermal: $\beta^{-1}\dot{\Sigma}$ is a dissipation/budget power measure). In the linear response regime one has the causal convolution

$$J_\alpha(t) \;=\; \sum_\beta \int_0^\infty \chi_{\alpha\beta}(s)\, f_\beta(t-s)\, ds \;=\; \sum_\beta \int_{-\infty}^t \chi_{\alpha\beta}(t-t')\, f_\beta(t')\, dt'.$$

Cross-reference

For the decomposition $J = J^{\mathrm{rev}} + J^{\mathrm{irr}}$ see Formula Box VIII.4.2.1. For $\dot{\Sigma} \geq 0$ and the Clausius balance see Lemma VIII.6.2.1 and Formula Box VIII.6.1.1.

By construction, the response kernels $\chi_{\alpha\beta}$ are causal. Under equilibrium/stationarity, $\chi_{\alpha\beta}$ depends only on the time difference; then time- and frequency-domain descriptions are

[40] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 (calibration & thermal scales).
[41] See FBA Part IV: Dynamics & Measurement (GKLS), Sec. IV.7 (stationary states & detailed balance).

equivalent, and transport coefficients can be identified as integrals of equilibrium correlators. This sets the stage for FDT/Green–Kubo: *response* becomes *fluctuation*.

VIII.9.2 FDT (time and frequency domain)

The fluctuation–dissipation relation links linear response to equilibrium fluctuations of the corresponding irreversible fluxes.[28, 29] Conceptually, this is particularly transparent in the FBA: the same currents J_α that generate dissipation (and thus irreversible budget) in the linear regime are measurable as fluctuations in equilibrium. Under microreversibility (local detailed balance/ldb), this connection becomes a precise identity.

Formula Box VIII.9.2.1: FDT I (time domain)

Let

$$C_{\alpha\beta}(t) \;:=\; \big\langle \delta J_\alpha(t)\, \delta J_\beta(0) \big\rangle_\star, \qquad \delta J_\alpha := J_\alpha - \langle J_\alpha \rangle_\star,$$

be the equilibrium correlator in the stationary reference measure $p^\star$ (notation $\langle\cdot\rangle_\star$). Here $\vartheta(t)$ denotes the Heaviside step function (causality). Under microreversibility and for the force–flux pairing conjugate to f_β, one has (classical FDT limit)

$$\chi_{\alpha\beta}(t) \;=\; \beta\, \vartheta(t)\, C_{\alpha\beta}(t).$$

In particular, the static coefficients

$$L_{\alpha\beta} \;:=\; \int_0^\infty \chi_{\alpha\beta}(t)\, dt \;=\; \beta \int_0^\infty C_{\alpha\beta}(t)\, dt$$

are positive semidefinite (and thus $\dot{\Sigma} = f^\top L f \geq 0$ in the linear regime).

The Heaviside function $\vartheta(t)$ encodes causality (no response before the drive). In frequency space this becomes the usual relation between spectra and dissipative (real) parts of the response; this form is often the direct interface to data analysis (spectral estimators, transfer functions).

Formula Box VIII.9.2.2: FDT II (spectral form)

With spectral density

$$S_{\alpha\beta}(\omega) \;:=\; \int_{-\infty}^\infty e^{i\omega t} C_{\alpha\beta}(t)\, dt$$

and dynamic susceptibility

$$\tilde{\chi}_{\alpha\beta}(\omega) \;:=\; \int_0^\infty e^{i\omega t} \chi_{\alpha\beta}(t)\, dt$$

one has in equilibrium (classical; for the conjugate pairing as in Formula Box VIII.9.2.1)

$$S_{\alpha\beta}(\omega) \;=\; 2\, \beta^{-1}\, \mathrm{Re}\, \tilde{\chi}_{\alpha\beta}(\omega) \;=\; 2\, k_B T\, \mathrm{Re}\, \tilde{\chi}_{\alpha\beta}(\omega).$$

VIII.9.3 Green–Kubo relations & Onsager–Casimir

The Green–Kubo formulas are the "measurement form" of linear response: transport coefficients can be extracted from equilibrium data (correlators).[28, 30] In the FBA this is simultaneously a budget statement: $L_{\alpha\beta}$ parametrizes how much irreversible internal budget accrues for given forces. The Onsager–Casimir symmetries specify which symmetries one may expect for L once time-reversal parities (and possibly external fields) are known.[31–33]

Formula Box VIII.9.3.1: Green–Kubo (static)

The (static) transport coefficients

$$L_{\alpha\beta} := \int_0^\infty \chi_{\alpha\beta}(t)\, dt$$

satisfy (under the equilibrium assumptions of the FDT)

$$L_{\alpha\beta} = \beta \int_0^\infty \langle \delta J_\alpha(t)\, \delta J_\beta(0)\rangle_\star\, dt = \frac{\beta}{2} \int_{-\infty}^\infty \langle \delta J_\alpha(t)\, \delta J_\beta(0)\rangle_\star\, dt.$$

Lemma VIII.9.3.1: Onsager–Casimir symmetries

Let $\epsilon_\alpha \in \{+1, -1\}$ be the time-reversal parities of the fluxes J_α. Under microreversibility (ldb) one has

$$L_{\alpha\beta}(\mathbf{B}) = \epsilon_\alpha\, \epsilon_\beta\, L_{\beta\alpha}(-\mathbf{B}),$$

in particular for $\mathbf{B} = 0$: $L_{\alpha\beta} = \epsilon_\alpha\, \epsilon_\beta\, L_{\beta\alpha}$.

Proof Sketch VIII.9.3.1: Onsager–Casimir symmetries

Time reversal Θ (parity as in Definition VIII.7.1.1) with $\Theta J_\alpha \Theta^{-1} = \epsilon_\alpha J_\alpha$ and ldb imply for equilibrium correlators

$$C_{\alpha\beta}(t; \mathbf{B}) = \epsilon_\alpha \epsilon_\beta\, C_{\beta\alpha}(t; -\mathbf{B}).$$

Integration in Formula Box VIII.9.3.1 yields the claim.

VIII.9.4 Positivity, bounds & budget interpretation

In the linear regime, entropy production becomes a quadratic form in the forces. This is the fastest route to robust bounds: positivity of $\dot{\Sigma}$ *for all* small drivings enforces $L \succeq 0$. Thus "negative dissipation" or unstable transport matrices in equilibrium are ruled out, independently of microscopic details.[28, 31]

Corollary VIII.9.4.1: Positivity of the transport matrix & second law

The matrix $L = [L_{\alpha\beta}]$ is positive semidefinite. For arbitrary (small) forces f,

$$\dot{\Sigma}(t) \;=\; \sum_{\alpha\beta} f_\alpha(t)\, L_{\alpha\beta}\, f_\beta(t) \;\geq\; 0.$$

Hence the linear dissipation $f^\top L f$ is a *lower* bound on the irreversible budget rate in the linear regime (consistent with Section VIII.6).

Experimental observables & budget flows

Conductivities/viscosities are integrals of equilibrium fluctuations—measurable via temporal correlation functions. In the FBA, they are simultaneously rates of irreversible *internal* budget usage per force: "dissipation per force squared". Calibration with β (temperature) as in [a].

[a]See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 (thermal scales).

VIII.9.5 Example: diffusion, mobility & Einstein relation

As a minimal case, Brownian motion demonstrates the link between (i) response to a small force field, (ii) equilibrium fluctuations, and (iii) thermal calibration via $k_B T$.[28, 34] This case is particularly useful as a "sanity check" for data analysis: it tests both the correlator estimation and the correct temperature calibration.

Einstein relation $D = \mu\, k_B T$ (underdamped)

Consider a 1D underdamped Langevin system in equilibrium at temperature T,

$$m\,\dot{v}(t) = -\gamma\, v(t) + f + \sqrt{2\gamma\, k_B T}\,\xi(t), \qquad \dot{x}(t) = v(t),$$

with $\langle \xi(t)\rangle = 0$, $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, and a small constant force field f.

1. **Mobility (definition).** In the stationary linear regime,

$$\langle v\rangle \;=\; \mu\, f, \qquad \mu = \gamma^{-1}.$$

2. **Diffusion (Green–Kubo).** In equilibrium ($f = 0$),

$$C_{vv}(t) := \langle v(t)\, v(0)\rangle_\star = \frac{k_B T}{m}\, e^{-(\gamma/m)t}, \qquad D = \int_0^\infty C_{vv}(t)\, dt = \frac{k_B T}{\gamma}.$$

3. **Einstein.** Hence,

$$D \;=\; \mu\, k_B T \;=\; \mu\, \beta^{-1}.$$

VIII.9.6 Edge cases: NESS & frequency-dependent response

Away from strict equilibrium, many structural elements remain (causality, Green–Kubo-type integrals), but symmetries and identities are *corrected*: stationary currents generate additional source terms, and the simple FDT identity must be replaced by NESS versions.[16, 35] This matters in applications because real systems are often operated in stationary nonequilibrium settings (gradients, drives, active baths).

Outside equilibrium

Outside strict equilibrium, *causality* and many *structural formulas* remain, but the simple equilibrium identities (FDT/Green–Kubo without additional terms) are generally corrected:

1. **NESS (stationary nonequilibrium).** For stationary nonequilibrium states, $p^\star$ replaces the equilibrium measure. Green–Kubo relations typically retain the "correlator integral" form, but acquire *additional source/violation terms* due to stationary currents (or deviations from the equilibrium FDT); see also the NESS path perspective in **??**.
2. **Dynamic response (frequency domain).** Formula Box VIII.9.2.2 gives the relation between spectra and $\operatorname{Re}\tilde{\chi}_{\alpha\beta}(\omega)$ in equilibrium. Independently, *causality* implies Kramers–Kronig relations linking the real and imaginary parts of $\tilde{\chi}_{\alpha\beta}(\omega)$.[36]

VIII.9.7 Checklist & classification

The following points collect testable consequences in the linear regime. They are phrased deliberately as data checks: correlator integrals, symmetry tests ($\mathbf{B} \to -\mathbf{B}$), and positivity tests for L.

Pass/Fail – transport in the FBA

- **FDT:** $\chi_{\alpha\beta}(t) = \beta\,\vartheta(t)\,C_{\alpha\beta}(t)$ (Formula Box VIII.9.2.1); spectral form Formula Box VIII.9.2.2.
- **Green–Kubo:** $L_{\alpha\beta} = \beta \int_0^\infty C_{\alpha\beta}(t)\,dt$ (Formula Box VIII.9.3.1); $L \succeq 0$.
- **Onsager–Casimir:** $L_{\alpha\beta}(\mathbf{B}) = \epsilon_\alpha \epsilon_\beta\, L_{\beta\alpha}(-\mathbf{B})$ (Lemma VIII.9.3.1).
- **Budget interpretation:** $\dot{\Sigma} = f^\top L f$ is the linear dissipative power (irreversible internal budget per time) in the linear regime.

VIII.9.8 Outlook

The Green–Kubo/FDT structure ties transport coefficients directly to budget fluctuations and makes dissipation accessible as a "correlator observable". Section VIII.10 uses these relations for predictions, falsifiability, and the design of measurement protocols (among others, linear-response tests & scaling laws; [42]).

[42] See FBA Part X: Predictions, Falsifiability & Bridge FBA $\rightarrow$ QM $\leftrightarrow$ GR, Sec. X.2 (linear response & test quantities).

VIII.10 Comparison & alignment with the standard

This Section places the results developed in Sections Section VIII.3 through Section VIII.9 into established frameworks (classical mechanics, decoherence/semiclassical limit, stochastic thermodynamics, linear response) and marks precisely where the FBA demands *more* (additional monotonicities/bounds, observable "aging") or presupposes *less* (no postulates about the classical world). For the formal starting blocks (CPTP/GKLS, DPI/Spohn, composition/locality) see [43] as well as the standard sources [3, 5–7]. For the development of measurement/decoherence/GKLS (incl. detailed balance) cf. [44] and [45]. Short bridges to the other parts of the series provide context without creating a jungle of references.

VIII.10.1 Bridges: concept correspondences

As a starting point, a "dictionary" level suffices: Which objects correspond to which standard notions? What matters is: the mapping is meant *operationally* (measurement and protocol perspective), not merely formally/symbolically. The pairings below therefore also indicate *where* in the text the respective structure is derived.

FBA ↔ standard (selection)

- *Budget calculus (internal/external/irreversible)* ↔ work/heat/entropy flows (Clausius), stochastic thermodynamics (Section VIII.6; Formula Box VIII.6.1.1) [16].
- *Pointer projection* $\mathcal{R}$ ↔ decoherence/POVM instruments; classical master limit (Section VIII.3; Corollary VIII.3.2.1) [1, 2].
- *Master → FP/Langevin* ↔ Kramers–Moyal/Fokker–Planck/Itô SDE (Section VIII.4; Lemma VIII.4.2.1 and Formula Boxes VIII.4.2.1 and VIII.4.3.1) [13–15, 37].
- *Reversible budget component* ↔ Liouville/Hamilton–Jacobi dynamics (Section VIII.5; Formula Boxes VIII.5.2.1 and VIII.5.2.2).
- *DPI/Spohn* ↔ H-theorem/second law (Section VIII.6; Lemma VIII.6.2.1 and Corollary VIII.6.2.1) [3, 5].
- *Path budget/EP* ↔ Crooks/Jarzynski (stochastic thermodynamics on path spaces) (Section VIII.7; Formula Boxes VIII.7.3.1 and VIII.7.4.1) [16, 21, 22].
- *Aging* A ↔ integrated *irreversible internal* budget component, calibrated isothermally via $\beta^{-1}\Sigma_{\text{int}}$ and converted to time via κ_τ (Section VIII.8; Formula Box VIII.8.2.1).
- *Green–Kubo/FDT* ↔ linear response/transport coefficients (Section VIII.9; Formula Boxes VIII.9.2.1 and VIII.9.3.1) [28–33].

These correspondences already explain why many results feel "familiar": the FBA reproduces standard structures, but motivates them via *admissible processing* (CPTP/GKLS) and

[43] See FBA Part I: FBA – Foundations, Secs. I.5–I.6 "Admissible dynamics, DPI/Spohn, composition".
[44] See FBA Part III: Quantum Kinematics & CPTP Channels, Secs. III.3–III.5 "POVMs, instruments & Naimark".
[45] See FBA Part IV: Dynamics & Measurement (GKLS), Secs. IV.4–IV.7 "Decoherence, entropy flow & stationary states".

monotonicities (DPI/Spohn) rather than via separate classical postulates. The next step is therefore not another "cross-check", but a clear statement of *where* the FBA enforces additional, testable structure.

VIII.10.2 Delimitation & added value relative to standard approaches

The points below summarize in what sense the FBA is *stronger* than standard presentations (more structure/bounds) and in what sense it is *more economical* (fewer postulates). Here "added value" is not meant as new phenomenology, but as: clearer logic paths, sharper bounds, and additional observable quantities.

Added value of the FBA (brief)

- **Deductive classical limit:** Classical equations arise from CPTP/GKLS + $\mathcal{R}$ + timescales—without extra postulates (Sections VIII.3 and VIII.4) [1, 2].
- **Monotonicities & no-recovery:** DPI/Spohn yields universal bounds for unselected processes; selective interventions are explicitly excluded (Section VIII.6; Lemma VIII.6.2.1) [3–5]. The boundary "unselected vs. selective" is central; cf. [a].
- **Budget interpretation of fluctuation theorems:** $\Sigma = \beta(W - \Delta F)$ links work estimates directly to *internal* irreversible budget (Section VIII.7; Formula Box VIII.7.2.1) [16, 21, 22]; conversion to aging time proceeds via κ_τ (Formula Box VIII.8.2.1).
- **Observable aging:** A separates dissipative lifetime from mere proper time—an observable beyond standard kinematics (Section VIII.8; Lemma VIII.8.1.1 and Formula Box VIII.8.2.1).
- **Transport as budget fluctuation:** FDT/Green–Kubo appear as positivity/symmetry statements about irreversible budgets (Section VIII.9; Corollary VIII.9.4.1 and Lemma VIII.9.3.1) [28–33].

[a] See FBA Part III: Quantum Kinematics & CPTP Channels, Secs. III.4–III.5 "Instruments & selective operations".

The central dividing line running through all points is the same as in the preceding Sections: *unselected admissible processing* (CPTP/GKLS, DPI/Spohn) produces robust monotonicities; anything that leaves this admissible class (postselection/feedback without bookkeeping) can generate local "counterexamples", but then this is not a core violation—it is a protocol change. This brings us to the question: *Which* statements are the minimal "main switches" (pass/fail) of this treatise?

VIII.10.3 Concrete, falsifiable statements (this treatise)

The statements below are phrased so that they serve as minimal *falsifiers*: each links a clear premise (pointer-stable, unselected, ldb, linear regime) to an unambiguously testable criterion. The complete, coherent pass/fail list is in Section VIII.11; here we focus on the most important "main switches". For protocol-side operationalization (how does one measure

J, W, Q, ΔF, A in minimally invasive setups?) see especially [46].

Formula Box VIII.10.3.1: Testable core statements

1. **Contraction after projection:** For pointer-stable $\mathcal{R}$, $D(p_t \| p^\star)$ decreases monotonically (Lemma VIII.3.2.1) [3, 4]. *Fail* if, in a verified *unselected* setup (and with a consistent reference choice $p^\star$), reproducible Markov marginals with $\dot{D} > 0$ are observed (beyond controlled $\mathcal{O}(\varepsilon)$ closure errors).

2. **Landauer in the budget picture:** A reset by ΔI requires in the isothermal limit
$$A \;\geq\; \frac{\beta^{-1}}{\kappa_\tau}\,\Delta I$$
(Corollary VIII.6.3.1 and Formula Box VIII.8.2.1) [19, 20]. *Fail* if reproducibly $A < (\beta^{-1}/\kappa_\tau)\Delta I$ occurs under controlled temperature calibration and without hidden selection/feedback.

3. **Crooks crossing:** $P_F(W) = P_R(-W)$ at $W = \Delta F$ (Section VIII.7; Formula Box VIII.7.3.1) [21, 22]. *Fail* if systematically shifted *given* ldb and correctly implemented time reversal (protocol reversal).

4. **FDT/Green–Kubo positivity:** $L \succeq 0$, $\dot{\Sigma} = f^\top L f \geq 0$ (Section VIII.9; Formula Boxes VIII.9.2.1 and VIII.9.3.1 and Corollary VIII.9.4.1) [28–33]. *Fail* if negative eigenvalues of L occur *in the linear regime* and *under microreversibility* (not merely a numerical/finite-size artifact).

5. **Separation τ_{geo} vs. A:** Varying kinematics (time dilation) at fixed, locally identically realized internal GKLS coupling does not change A (Section VIII.8; Lemma VIII.8.4.1). *Fail* if A scales systematically with pure kinematics without any change of the internal dissipation channels.

One may also read this list as "minimal diagnostics": (1)–(2) test DPI/Spohn and budget/temperature calibration in the thermal limit, (3) tests path reversal/consistency of ldb, (4) tests the linear-response structure as a correlator observable, (5) tests the new conceptual separation between geometric time and dissipative aging. Next it is therefore worth taking a quick look at the *document bridges*: Which blocks are curved/renormalized/continued in other parts?

VIII.10.4 Quick bridge FBA $\to$ QM $\leftrightarrow$ GR

The statements developed in this treatise are kinematic/flat. The structural bridges nevertheless remain visible, because the building blocks used here (calibration, GKLS/DPI, budget flows) are precisely those that are further developed in the other parts. [47] [48] [49]

[46] See FBA Part X: Predictions, Falsifiability & Bridge FBA $\to$ QM $\leftrightarrow$ GR, Sec. X.2 "Test quantities & protocols".

[47] See FBA Part II: Time, Proper Time & Minkowski Geometry, (i) proper-time geometry as a Minkowski limit.

[48] See FBA Part VI: Gravity & Geometry from Budget Flows, (ii) geometrization of budget flows.

[49] See FBA Part VII: Constants, Scales & Renormalization, (iii) calibration/RG/scale windows.

Bridges (brief, with references)

- **Proper time/Minkowski:** τ_{geo} follows from reversible budget (Minkowski quadric/light cone/Lorentz limit), and aging A adds as an irreversible internal component ([a]; [b]).
- **Geometry from budget flows:** In curved situations, budget flows are geometrized; this treatise provides the flat thermo/transport basis ([c]).
- **Scales/RG:** Normalizations (β, c, k_B) and response scales are made precise in Part VII—here they are only used ([d]).

[a] See FBA Part II: Time, Proper Time & Minkowski Geometry, Sec. II.4 "Proper time & aging".

[b] See FBA Part II: Time, Proper Time & Minkowski Geometry, Secs. II.6–II.7 "Minkowski limit & Lorentz kinematics".

[c] See FBA Part VI: Gravity & Geometry from Budget Flows, Secs. VI.2–VI.4 "Budget geometry".

[d] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.3 "Calibration & response scales".

This bridge is also a hint for interpretation: If apparent deviations show up in experiment/simulation, the first check is often whether an assumption of this treatise has been violated (e. g. selectivity, missing timescale separation, NESS instead of equilibrium) or whether a core criterion (monotonicity/bound) is actually broken. We therefore close with an explicit "assumptions box" as a troubleshooting tree.

VIII.10.5 Conditions & domain of validity

The strength of the results rests on clear premises. We name them explicitly here because they also mark the standard sources of "apparent counterexamples": one (deliberately or inadvertently) leaves the admissible domain, and then violations of monotonicities are not surprising.

Assumptions (explicit) & typical failure modes

- **Pointer stability & secular limit:** Off-diagonals relax faster than observation times (Section VIII.3); cf. [a] and [1, 2].
- **Unselected GKLS dynamics:** Monotonicities/bounds hold without postselection (Sections VIII.6 and VIII.7) [5–7]. Selective instruments/feedback are explicitly excluded; cf. [b].
- **Diffusive scaling limit:** For FP/Langevin (Section VIII.4) one needs finite second moments and small jumps [13–15].
- **Linear response regime:** FDT/Green–Kubo hold near stationary references (Section VIII.9), typically under microreversibility/detailed balance; cf. [c] as well as [28–33].
- **Flat/kinematic setting:** No backreaction/curvature; for gravitation see [d].

[a] See FBA Part IV: Dynamics & Measurement (GKLS), Secs. IV.4–IV.6 “Decoherence, secular limit & effective jumps”.

[b] See FBA Part III: Quantum Kinematics & CPTP Channels, Secs. III.4–III.5 “Instruments & selective operations”.

[c] See FBA Part IV: Dynamics & Measurement (GKLS), Sec. IV.7 “Stationary states & detailed balance”.

[d] See FBA Part VI: Gravity & Geometry from Budget Flows, Part VI.

VIII.10.6 Classification & outlook

The FBA perspective ties the classical limit, thermodynamics, and transport to a single monotonicity principle (DPI/Spohn) and to an operational budget calculus. Section VIII.11 consolidates the pass/fail criteria of this treatise and provides a compact checklist for peer review, simulations, and experiments.

VIII.11 Summary & checklist (Pass/Fail)

This Section collects the central statements of the treatise (Section VIII.3 to Section VIII.10) and formulates a compact, operationally usable pass/fail checklist. The derived building blocks are: decoherence/pointer & master-limit (Section VIII.3), Kramers–Moyal/FP/Langevin (Section VIII.4), Ehrenfest/Hamilton limit (Section VIII.5), entropy production/2nd law/-Landauer (Section VIII.6), fluctuation theorems (Section VIII.7), aging (Section VIII.8), transport/FDT/Green–Kubo (Section VIII.9), positioning (Section VIII.10). For measurement-practice protocols/design and bridge statements see also [50]. The criteria are chosen such that they map to directly measurable quantities (population/current data, work/heat, correlators) and test the monotonicities, relations, and separations derived in the preceding Sections.

VIII.11.1 Brief conclusion of the treatise

We first summarize the "load-bearing pillars", because the checklist operationalizes exactly these structural blocks as measurement criteria: (1) projection/closure ⇒ classical dynamics, (2) reversible vs. irreversible parts ⇒ trajectories/EP, (3) path level ⇒ Crooks/Jarzynski, (4) new observable quantity ⇒ aging, (5) linear response ⇒ transport as a correlator observable.

Core statements (compact)

- **Classical limit:** pointer-stable projection $\mathcal{R}$ and time-scale separation lead from GKLS to an autonomous master equation (Corollary VIII.3.2.1); the continuous limit yields Fokker–Planck/Langevin (Lemma VIII.4.2.1 and Formula Boxes VIII.4.2.1 and VIII.4.3.1).
- **Reversible dynamics:** the reversible budget component yields Liouville/Hamilton–Jacobi structure and effective trajectories (Formula Boxes VIII.5.2.1 and VIII.5.2.2 and Lemma VIII.5.3.1).
- **Second law:** entropy production $\dot{\Sigma} \geq 0$ from Spohn/DPI and the Clausius balance (Lemma VIII.6.2.1, Formula Box VIII.6.1.1, and Corollary VIII.6.2.1); Landauer bound in the budget picture (Corollary VIII.6.3.1).
- **Fluctuation theorems:** IFT, Crooks, and Jarzynski connect path-EP with work/free energy (Lemma VIII.7.2.1 and Formula Boxes VIII.7.2.1, VIII.7.3.1 and VIII.7.4.1).
- **Aging A:** integrated *irreversible internal* budget flow, operationally calibratable via EP (Definition VIII.8.1.1, Formula Box VIII.8.2.1, and Lemma VIII.8.4.1).
- **Transport:** FDT/Green–Kubo, positivity, and Onsager–Casimir as properties of irreversible budgets (Formula Boxes VIII.9.2.1, VIII.9.2.2 and VIII.9.3.1, Lemma VIII.9.3.1, and Corollary VIII.9.4.1).

VIII.11.2 Pass/fail checklist (operational)

The checklist organizes testable statements by measurement channel. "Pass" means: measured data lie within the specified tolerance band; "Fail": systematic, reproducible violation

[50] See FBA Part X: Predictions, Falsifiability & Bridge FBA → QM ↔ GR, Secs. X.1–X.3 "Measurement programs, protocols & checklists".

beyond uncertainties. What matters is the protocol boundary: monotonicities/bounds hold for *unselected* flows (no postselection/feedback trick), and path relations (Crooks/Jarzynski) assume microreversibility/ldb as well as consistent temperature calibration. For the selective/unselected distinction and instruments see [51]; for detailed-balance structure in the GKLS framework [52]; for temperature/units calibration [53].

Formula Box VIII.11.2.1: Test catalog with target quantities & tolerances (I)

(A) Projection & master dynamics. *Target quantity:* monotonicity of the classical relative entropy $D(p_t \| p^\star)$. *Criterion:* $\Delta D/\Delta t \leq \delta_A$ on unselected data traces (Lemma VIII.3.2.1 and Corollary VIII.3.2.1). *Pass:* fraction of positive slopes $< z_A$ (outlier rate). *Fail:* significant positive trends (not merely measurement noise/$\mathcal{O}(\varepsilon)$ closure errors).

(B) FP/Langevin limit. *Target quantity:* positive semidefiniteness of the diffusion $D(x,t)$ from data reconstruction. *Criterion:* all empirical eigenvalues $\lambda_i(D) \geq -\delta_D$ (Lemma VIII.4.2.1 and Formula Box VIII.4.2.1). *Pass:* $\max_i[-\min(0,\lambda_i)] \leq \delta_D$. *Fail:* robust negativity across windows/bootstrap.

(C) Second law & Landauer/aging. *Target quantity:* $\dot{\Sigma} \geq 0$ (unselected) and the Landauer bound in the aging measure. *Criterion:* time-integrated $\Sigma \geq -\delta_\Sigma$ (Lemma VIII.6.2.1 and Corollary VIII.6.2.1); reset campaigns:

$$A - \frac{\beta^{-1}}{\kappa_T}\Delta I \;\geq\; -\delta_{\text{Lan}} \qquad (Corollary\ VIII.6.3.1 and Formula Box\ VIII.8.2.1).$$

Pass: both bounds satisfied. *Fail:* reproducible violation under controlled calibration (β, κ_T) and an unselected setup.

[51] See FBA Part III: Quantum Kinematics & CPTP Channels, Secs. III.4–III.5 “Instruments & selective operations”.

[52] See FBA Part IV: Dynamics & Measurement (GKLS), Sec. IV.7 “Stationary states & detailed balance”.

[53] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 “Calibration & thermal scales”.

Formula Box VIII.11.2.2: Test catalog with target quantities & tolerances (II)

(D) Fluctuation theorems. *Target quantity:* Crooks symmetry and Jarzynski moment. *Criterion:* $\log \frac{P_F(W)}{P_R(-W)}$ vs. W linear with slope β and intercept at $W = \Delta F$ (Formula Box VIII.7.3.1); $\langle e^{-\beta W}\rangle = e^{-\beta \Delta F}$ within δ_J (Formula Box VIII.7.4.1). *Pass:* both satisfied. *Fail:* systematic deviation *given* microreversibility/ldb (time-reversal protocol correctly implemented).

(E) Aging vs. proper time. *Target quantity:* separation of A from τ_{geo}. *Criterion:* comparison-clock protocols yield $\Delta A \neq 0$ at constant τ_{geo} and vice versa (Lemma VIII.8.4.1 and Subsection VIII.8.5). *Pass:* expected separation visible. *Fail:* A scales primarily with kinematics (with unchanged internal dissipation).

(F) Transport/response. *Target quantity:* $L = \beta \int_0^\infty C\,dt$ positive semidef.; Onsager–Casimir symmetry. *Criterion:* $\text{mineig}(L) \geq -\delta_{\text{pos}}$ (Formula Box VIII.9.3.1 and Corollary VIII.9.4.1); $L_{\alpha\beta}(\mathbf{B}) \approx \epsilon_\alpha \epsilon_\beta L_{\beta\alpha}(-\mathbf{B})$ within δ_O (Lemma VIII.9.3.1). *Pass:* both satisfied. *Fail:* robust violation (not merely a finite-size/finite-window artifact).

VIII.11.3 Measured quantities, data acquisition & analysis

To make the checklist truly "plug-in", we collect here, for each test item, the minimally required observables. The idea is: each block is (i) accessible via a standard measurement and (ii) translatable via a standard analysis into a pass/fail criterion (with bootstrap/window checks for robustness).

Minimal dataset per test item

- **(A)** Time series p_t, estimated $p^\star$, relative-entropy trace $D(p_t \| p^\star)$.
- **(B)** Drift/diffusion reconstruction from short-lag Δt increments; eigenvalue test of $D(x,t)$.
- **(C)** $\dot{S}_{\text{sys}}$, heat fluxes $\dot{Q}_\alpha$, $\dot{\Sigma}$; reset experiments with ΔI and calibration β and κ_τ.
- **(D)** Work samples W for forward/reverse protocols; ΔF from an equation of state or an independent reference measurement.
- **(E)** Comparison-clock traces: τ_{geo} (kinematics) and internally booked $\dot{\Sigma}_{\text{int}}$ to determine A (Formula Box VIII.8.2.1).
- **(F)** Equilibrium correlators $C_{\alpha\beta}(t)$; response functions $\chi_{\alpha\beta}(t)$; if applicable, **B**-dependent measurement.

Analysis & uncertainties

- **Relative entropy $D(p_t \| p^\star)$:** sensitive for rare states/$p \approx 0$; regularization (floor/capping of small p) and bootstrap intervals recommended.
- **Diffusion/drift (FP reconstruction):** short lags Δt, drift subtraction, local quadratics; check window choice and stability under Δt variation.
- **Jarzynski ($\langle e^{-\beta W} \rangle$):** exponential reweighting ⇒ variance-sensitive; variance reduction via Bennett acceptance ratio/stratification (cf. **??**).
- **Transport (Green–Kubo):** window choice for $\int_0^\infty C(t)\,dt$, tail fits/decay models; outside equilibrium include additional source terms/stationary currents.

VIII.11.4 Fail analysis & diagnostics

A "fail" is informative only if it is cleanly attributed to the right cause: (i) assumptions/protocols violated, (ii) reconstruction/analysis unstable, or (iii) an actual structural break. The following heuristic is intended as a quick diagnostic tree (first protocol/assumptions, then analysis, then interpretation).

Typical causes of "fail" (heuristic)

(A) Violated pointer stability/sector closure: ⇒ refine $\{\Pi_x\}$, ensure a larger τ_{dec} gap, verify unselected data acquisition.

(B) Nondiffusive limit (fat jumps): ⇒ model jump terms of 3rd and higher order instead of FP; if needed, stay at the master-equation level.

(C) Selective/feedback-driven protocols: ⇒ monotonicities hold only unselected (Lemma VIII.6.2.1); explicitly account for instruments/selection (protocol change).

(D) ldb violated (non-isothermality, multiple baths, driving too fast): ⇒ use NESS/multi-bath forms (cf. Section VIII.7 and **??**).

(E) Wrong calibration of β or κ_τ, inconsistent ΔF reference: ⇒ check thermal scales, time calibration, and reference measurement; cf. [a].

(F) Transport: finite-window/finite-size effects: ⇒ tail fits, block bootstrap, test $\mathbf{B} \to -\mathbf{B}$ symmetry separately (cf. Section VIII.9).

[a] See FBA Part VII: Constants, Scales & Renormalization, Secs. VII.1–VII.2 "thermal scales".

VIII.11.5 Closing remark

The checklist is modular: each test item stands on its own and can be verified independently. Together they form a coherent empirical validation of the FBA picture in the classical limit—from the projection through thermodynamics/fluctuations to aging and transport. Implementation on concrete platforms (quantum optics, condensed matter, soft matter) as well as protocol design and scaling laws are developed further in *Part X*; cf. [54] for concrete measurement paths and robust estimators.

[54]See FBA Part X: Predictions, Falsifiability & Bridge FBA $\to$ QM $\leftrightarrow$ GR, Sec. X.2 “Test quantities & protocols”.

VIII.12 Appendix: Overview of the FBA Series (Parts I–X)

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1. ***Part I*: FBA-Foundations: Ordering, Budget, Proper Time & Arrows.** *Goal:* Provide the base layer: ordering, budget, proper time/aging, front and the operational arrow of time (DPI); Minkowski limit from the budget quadric; admissible dynamics and locality/no-signalling. *Import:* – (reference for all subsequent parts). *Extension:* interface contracts, pass/fail checklists, reading guide.
2. ***Part II*: Time, Proper Time & Minkowski Geometry.** *Goal:* Capture proper time/quadric operationally and derive geodesics. *Import:* foundations (ordering, budget, proper time, front/DPI). *Extension:* smooth limit, variational principle on worldlines, calibration κ_τ.
3. ***Part III*: Quantum Kinematics & CPTP Channels.** *Goal:* State spaces and channels (CPTP) including composition. *Import:* foundations (budget, channel viewpoint, composition). *Extension:* concrete divergences/cost functionals $\mathcal{C}$, measurements, and classical registers.
4. ***Part IV*: Dynamics, Measurement & GKLS (Open Systems).** *Goal:* Continuous open dynamics (GKLS) and the operational arrow of time. *Import:* channels/DPI. *Extension:* Spohn monotonicity, stationary/NESS references, flows $b^{\mathrm{rev}}, b^{\mathrm{irr}}, b^{\mathrm{ext}}$.
5. ***Part V*: Spacetime, Light Cones & Local Field Theory.** *Goal:* Local field equations under front/locality. *Import:* front, composition, no-signalling. *Extension:* local GKLS generators, Lieb–Robinson-type bounds, effective light cones.
6. ***Part VI*: Gravity & Geometry from Budget Flows.** *Goal:* Geometrization of budget flows. *Import:* budget quadric/proper time. *Extension:* effective metrics from calibrations (κ_t, κ_x) and internal stresses; coupling to curvature.
7. ***Part VII*: Constants, Scales & Renormalization.** *Goal:* Scale running of the calibration theorems. *Import:* $c = \kappa_t/\kappa_x$, κ_τ. *Extension:* flow equations for $\kappa_t, \kappa_x, \kappa_\tau$; stability of c.
8. ***Part VIII*: Classical Limit, Thermodynamics & Aging.** *Goal:* Macroscopic behavior from $A[\gamma]$ (aging) and DPI. *Import:* proper time/aging, Spohn. *Extension:* entropy production, Euler–Lagrange forms for irreversible flows, effective transport equations.
9. ***Part IX*: Cosmic Dynamics, Time Dilation & Inflation (TDI).** *Goal:* Cosmic ordering & calibration flow. *Import:* budget, proper time/front. *Extension:* budget equations on large-scale slices; time-dilation inflation as calibration dynamics.
10. ***Part X*: Predictions, Falsifiability & Bridge FBA → QM ↔ GR.** *Goal:* Testable differences and bridges FBA ↔ QM/GR. *Import:* all foundational building blocks. *Extension:* protocols, limiting-case tests, overdetermined consistency relations (pass/fail).

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